Transcendental meromorphic functions with three singular values

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Abstract

Every transcendental meromorphic function f in the plane, which has only three critical values, satisfies

$$\liminf_{r\to\infty} \frac{T(r,f)}{\log^2 r} \geq \frac{\sqrt{3}}{\pi},$$

and this estimate is best possible.

A *singular value* of a meromorphic function f in the plane $\mathbf{C}$ is, by definition, a critical value or an asymptotic value. If we denote the closure of the set of singular values by S then

$$f : \mathbf{C}\backslash f^{-1}(S) \to \overline{\mathbf{C}}\backslash S$$

is a covering map. Meromorphic functions with finitely many singular values play an important role in the value distribution theory (see, for example, [8, 15, 16], and Jim Langley's papers on the distribution of values of derivatives), as well as in holomorphic dynamics [4, 6].

In this paper, "meromorphic function" will always mean a *transcendental*[1] *meromorphic functions in the plane*, unless some other region is specified.

Langley [11, 12] discovered that there exists a relation between the number of singular values of a meromorphic function f and the growth of the

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[1]On algebraic functions with three critical values see [3, 9].

Nevanlinna characteristic $T(r,f)$. In [12] he proved that all meromorphic functions f with finitely many singular values satisfy

$$\liminf_{r\to\infty}\frac{T(r,f)}{\log^2 r}>0.$$

On the other hand, he showed in [13] that for every $\epsilon>0$ there exists a meromorphic function f with four singular values, such that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\log^2 r}<\epsilon.$$

Concerning meromorphic functions with three singular values, Langley proved in [13] that they satisfy

$$\liminf_{r\to\infty}\frac{T(r,f)}{\log^2 r}\geq c, \tag{1}$$

where c is an absolute constant.

In this paper, the precise value of this constant is found. I thank Walter Bergweiler who brought [13] to my attention and suggested the extremal problem which the following theorem solves.

Theorem *Let f be a meromorphic function with at most three singular values. Then* (1) *holds with $c=\sqrt{3}/\pi$, and there exists a meromorphic function f_0 with three singular values, such that $T(r,f_0)/\log^2 r\to\sqrt{3}/\pi$ as $r\to\infty$.*

Remarks

1. If f has finitely many singular values and at least one of them is an asymptotic value, then

$$\liminf_{r\to\infty}\frac{T(r,f)}{\sqrt{r}}>0, \tag{2}$$

in particular this holds for all entire functions with finitely many singular values. We sketch a proof of this, different from that mentioned in [13]. If a is an asymptotic value, and there are no other singular values in an ϵ-neighborhood of a, then one of the components of the set $\{z:|f(z)-a|<\epsilon\}$ is an unbounded region D whose boundary consists of one simple curve, and $f(z)\neq a$ in this region. Applying the standard growth estimates to the harmonic function $\log|f(z)-a|^{-1}$ in D we conclude that (2) holds.

2. *Let f be a meromorphic function with at most three critical values. Then the conclusion of the Theorem holds.*

Indeed, we can assume that f has finite lower order (otherwise there is nothing to prove). For such function with finitely many critical values, the set of asymptotic values is also finite. This was proved in [5] for functions of finite order and extended in [10] to functions of finite lower order. If there are asymptotic values, we apply Remark 1, if there are none, we apply the Theorem.

Similar improvement can be made in the results of Langley mentioned above.

3. All meromorphic functions with two singular values are of the form $L \circ \exp$, where L is a fractional-linear transformation.

4. Let μ be a probability measure on the Riemann sphere $\overline{\mathbf{C}}$, and $\nu = f^*\mu$ the pull-back of μ. We denote

$$A_\mu(r) = \nu\left(\{z : |z| \leq r\}\right), \quad r > 0.$$

If μ is the normalized spherical area, then $A_\mu(r) \equiv A(r)$, is the average number of sheets of the map $f : \{z : |z| \leq r\} \to \overline{\mathbf{C}}$. The Nevanlinna characteristic satisfies

$$T(r,f) = \int_1^r \frac{A(t)}{t} dt + O(\log r), \quad r \to \infty.$$

For arbitrary probability measure μ on the sphere, we have

$$\int_1^r \frac{A_\mu(t)}{t} dt \leq T(r,f) + O(1),$$

which is a consequence of the First Fundamental Theorem of Nevanlinna, [14, VI,§4]. Thus, to prove our theorem, it is sufficient to show that

$$A_\mu(t) \geq \frac{\sqrt{3}}{\pi} \log t + O(1), \quad t \to \infty, \tag{3}$$

for some probability measure μ.

Proof of the Theorem.

Let F be a surface of finite topological type, possibly with the boundary. A *triangular net* on F is a locally finite covering of F by closed sets T called *triangles*, such that

(i) Each triangle T is a closed Jordan region (homeomorphic to a closed unit disc) with three marked distinct boundary points called *vertices*. A closed boundary arc between two adjacent vertices is called an *edge* of T.

(ii) The intersection of two triangles is either empty, or a union of common edges and common vertices.

(iii) All triangles are divided into two classes, white and black, so that any two triangles with a common edge are of different colors.

Let f be a meromorphic function satisfying the assumptions of the Theorem. In view of Remark 1 we assume without loss of generality that f has no asymptotic values. Composing with a fractional-linear transformation we reduce to the case that the critical values are $0, 1$ and ∞.

Consider the triangular net N_0 on the Riemann sphere, which consists of two triangles, the closed upper half-plane (white) and the closed lower half-plane (black), and vertices $0, 1$ and ∞.

We construct the f-preimage of this net, which will be called N. The vertices of N are preimages of the vertices of N_0. The triangles of N are closures of components of preimages of the open upper and lower half-planes, and the edges of N are defined in the evident way. Each triangle in N is assigned the same color as its image in N_0.

It is easy to see that N indeed satisfies the conditions (i)-(iii), and the set of triangles in N is infinite because f is transcendental.

Choose one triangle $T_0 \in N$ and remove its interior $\operatorname{int} T_0$ from the plane. We obtain a surface with the boundary $D = \mathbf{C} \backslash \operatorname{int} T_0$ which is homeomorphic to a closed half-infinite cylinder.

Let $G \subset D$ be a compact closed region homeomorphic to a closed ring, which separates ∂T_0 from ∞. We are going to estimate from below the *extremal length*[2] λ of the family of all closed non-contractible curves in G. For this purpose we construct a conformal metric ρ in D. In this metric, each triangle of the net $N \backslash T_0$ will be isometric to a Euclidean equilateral triangle Δ with sidelength 1.

First we define a flat conformal metric ρ_0 on $\overline{\mathbf{C}} \backslash \{0, 1, \infty\}$. Let g be the conformal map of Δ onto the upper half-plane sending the vertices of Δ to $0, 1, \infty$. (An explicit expression of g will be given at the end of the paper). The metric ρ_0 is defined in the upper half-plane by the length element $ds = |(g^{-1})'(z)|\,|dz|$, so that g becomes an isometry from Δ with the Euclidean

[2] See, for example, [2] for a definition and simplest properties.

metric to the upper half-plane with the metric ρ_0. Using the Symmetry Principle, we extend ρ_0 to $\overline{\mathbf{C}}\backslash\{0,1,\infty\}$.

The Riemann sphere with the metric ρ_0 can be visualized as a "two-sided triangle Δ". The area of the sphere with respect to ρ_0 is $\sqrt{3}/2$.

Now we define $\rho = f^*\rho_0$, the pull-back of ρ_0 via f.

Metrics ρ and ρ_0 have isolated singularities at the vertices, but this does not cause any problems.

Now we estimate the ρ-length of the shortest non-contractible curve in D from below. It is clear that a shortest curve γ exists.

Lemma *Let D be a topological ring with a triangular net equipped with an intrinsic metric[3] such that every triangle is isometric to Δ. Then the length of every non-contractible curve in D is at least $\sqrt{3}$.*

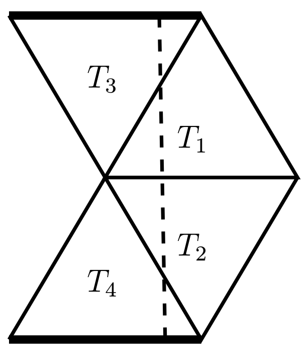


Figure 1. An extremal configuration. Bold segments are identified. An extremal curve is the broken line.

Proof. Let γ be a shortest non-contractible curve. Then γ is homeomorphic to a circle, because otherwise we could remove extra loops and shorten γ.

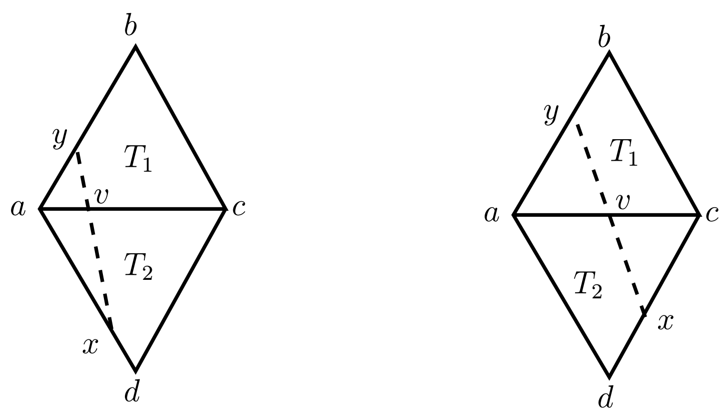
Suppose first that γ passes through a vertex v, that is $\gamma(t_0) = v$. Let F be the interior of the union of all triangles that have v as a common vertex. Then F is a simply connected region, and to be non-contractible, our curve has to pass trough a point $w \in \partial_D F = \partial F \cap \operatorname{int} D$, But then two arcs of γ from v to w have lengths at least $\sqrt{3}/2$ each, which proves the Lemma in this case.

[3] A metric is called intrinsic if the distance between any two points equals the infimum of lengths of curves connecting these points.

From now on we assume that γ does not pass through the vertices.

As our metric ρ is flat away from the vertices, every point $v \in D\backslash\{\text{vertices}\}$ has a neighborhood W which is isometric to a region V in the plane with the standard (intrinsic) metric. The isometry $\phi : V \to W$ has an analytic continuation to any simply connected region V' in the complement of a hexagonal lattice. In such a region, the map ϕ is a path-isometry, that is preserves the lengths of curves. Assuming that $v \in \gamma$, and $\phi(w) = v$, we can pull back γ to a curve passing through w. This pull-back will be a straight line in V'.

A union of two triangles with a common edge will be called a *rhombus*. It is path-isometric to a standard rhombus in the plane, as one of those shown on Figure 2. Evidently our curve γ intersects some edge. Let v be such a point of intersection, $\gamma(t_0) = v$. Consider the rhombus R_0 made of the two triangles T_1 and T_2 that have a common edge containing v (see Figure 2). Let $[t_1, t_2]$ be the maximal closed interval of parameter, containing v_0 and such that $\gamma(t) \in R_0$ for $t \in [t_1, t_2]$. Let γ_0 be the piece of γ from t_1 to t_2, and $x = \gamma(t_1)$, $y = \gamma(t_2)$. Then γ_0 is a straight line segment $[x, y]$ in a flat rhombus, and there are two cases to consider, depending of the way γ_0 intersects the rhombus (see Figure 2):



Case 1. Case 2.

Figure 2.

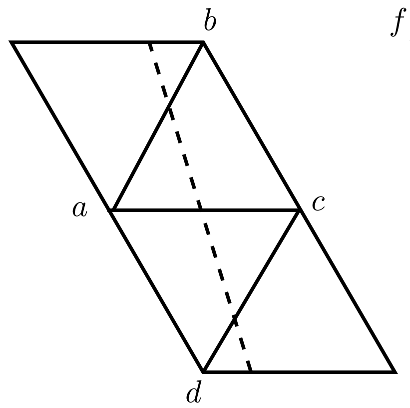
In Case 1, let T_3 be the triangle attached to T_1 along the edge (a, b).

We claim that $T_3 \neq T_2$. Indeed, suppose the contrary, which means that (a, b) is identified with (a, d). We assume without loss of generality that $|(a, y)| \leq |(a, x)|$. Let $y' \in (a, d)$ be the point such that $|(a, y)| = |(a, y')|$. We orient γ from x to y and the edge (a, b) from a to b. Then the angle between positive directions of γ_0 and (a, b) is at most $\pi/3$. The continuation of γ_0 after the point y is represented by a segment in R_0 which begins at y' and makes an angle at most $\pi/3$ with the edge (a, d) (directed from a to d). Then this segment has to cross γ_0 inside R_0, a contradiction. This proves our claim.

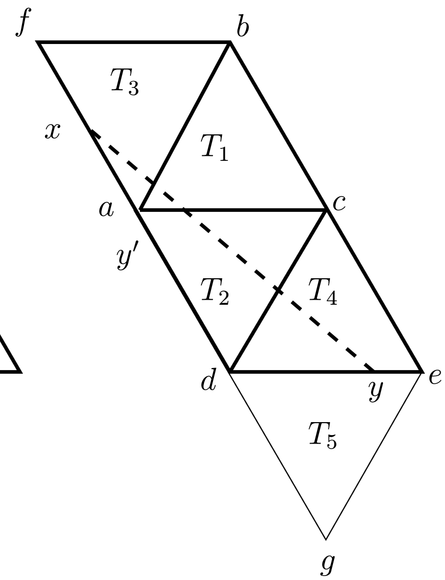
So T_1, T_2 and T_3 are three different triangles. Let T_4 be the triangle attached to T_2 along (a, d). Then $T_4 \neq T_1$ for the same reasons as $T_3 \neq T_2$, and in addition $T_3 \neq T_4$ because they are of different colors. So we have a picture as in Figure 1, and thus the length of γ_0 is at least $\sqrt{3}$. This completes the proof in Case 1.

In Case 2, we first notice that the edge (a, b) cannot be identified with the edge (d, c), because otherwise the edges (b, c) and (a, d) will be loops.

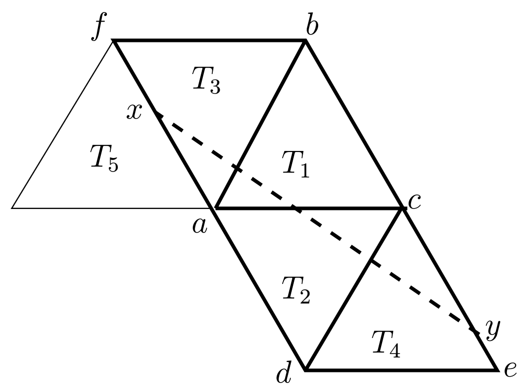
So the triangle T_3 attached to T_1 along (a, b) is different from T_2. For the similar reason, the triangle T_4 attached to T_2 along (d, c) is different from T_1. Furthermore, $T_3 \neq T_4$ because they are of different color. Thus we have a configuration of four different triangles making a “parallelogram” R_1 (see Figure 3). Let γ_1 be the maximal closed segment of γ which contains γ_0 and is contained in R_1. We consider three subcases, 2.1, 2.2 and 2.3 according to the position of γ_1 with respect to R_1.



Case 2.1



Case 2.2



Case 2.3

Figure 3.

In Case 2.1 we immediately conclude that the length of γ_1 is at least $\sqrt{3}$.

In Case 2.2, let T_5 be the triangle which is attached to T_4 along (d, e).

We claim that $T_5 \neq T_2$. Indeed, the contrary means that (d, a) is identified with (d, e). If we orient γ_1 from x to y (see Figure 3, Case 2.2) then γ makes an angle less than $\pi/3$ with (d, e), so the continuation of γ_1 after y will be represented by a segment in R_1 starting at $y' \in (d,a)$ and making an angle less than $\pi/3$ with (d,a). Then this continuation will cross γ_1, inside R_1 a contradiction which proves our claim.

Next we consider the possibility that $T_5 = T_3$. If this is so, then (d,e) is identified either with (a,f) or with (f,b). But (d,e) cannot be identified with (a,f) because this will make (a,d) a loop. If (d,e) is identified with (f,b), we consider the continuation of γ_1 to T_5 (which is identified with T_3) and easily conclude that the length of this this continued curve is at least $\sqrt{3}$, because the distance in $R_1 \cup T_5$ between (g,e) and (a,f) is $\sqrt{3}$.

The same argument applies if $T_5 \neq T_3$, which completes the proof in Case 2.2.

In the remaining Case 2.3, we first consider the possibility that (a,f) is identified with (e,c). Then γ_1 can be a closed curve. This happens if and only if $|(a,x)| = |(e,y)|$ and in this case the length of γ_1 is exactly $\sqrt{3}$. If γ_1 is not a closed curve then its continuation after the point y can be represented in R_1 by a segment parallel to γ_1 and connecting a point $x' \in (e,c)$ with a point $y' \in (a,f)$. The length of this segment, as well as the length of γ_1, is at least 1, so the total length of γ is at least 2 in this case.

So we may assume that (e,c) is not identified with (a,f). Consider the triangle T_5 adjacent to T_3 along (a,f). Then $T_5 \neq T_1$ (otherwise (f,b) will be a loop), and $T_5 \neq T_4$. Indeed, by assumption, (a,f) is not identified with (e,e) and (a,f) cannot be identified with (d,e) (otherwise (a,d) will be a loop). Furthermore, $T_5 \neq T_2$ because they are of different colors.

Thus we have five different triangles as on Figure Case 2.3, and it is clear that the distance between (e,c) and (g,f) in the union of these five triangles is $\sqrt{3}$. This completes the proof of the Lemma. $\square$

As a corollary we obtain that the extremal length λ of the set of non-contractible curves in any compact ring $G \subset D$ separating the ∂T_0 from ∞ in D satisfies

$$\lambda \geq \frac{3}{\text{area}\,(G)}, \tag{4}$$

where the area corresponds to the metric ρ.

To complete the proof, we consider the set

$$G(t) = \{z : |z| \leq t\} \backslash \text{int}\, K_0.$$

This set is homeomorphic to a ring if t is large enough. The extremal length $1/\lambda$ of the family of curves connecting the boundary components of the ring

$G(t)$ is $(2\pi)^{-1}\log t + O(1)$ as $t \to \infty$. According to (4), the area of this ring with respect to the metric ρ is at least

$$\frac{3}{\lambda} = \frac{3}{2\pi}\log t + O(1), \quad t \to \infty.$$

The ρ_0-area of the Riemann sphere is $\sqrt{3}/2$. Taking μ to be the ρ_0-area divided by $\sqrt{3}/2$, we obtain $A_\mu(t) \geq (\sqrt{3}/\pi)\log t + O(1)$, which is (3). $\square$

Example

Consider the equilateral triangle $\Delta \subset \mathbf{C}$ with vertices $0, i$ and $(\sqrt{3} + i)/2$. Let g be a conformal map of this triangle onto the right half-plane, sending the vertices to $\infty, ia, -ia$, where $a > 0$. By reflection, g extends to a meromorphic function in $\mathbf{C}$ with no asymptotic values and three critical values, $ia, -ia$ and ∞. All preimages of the critical values are critical points of order 3.

This function g is doubly periodic and its shortest period is $\sqrt{3}$. It can be expressed in terms of the Weierstrass function of an equiharmonic lattice, see [1, 7]. If we choose

$$a = k^3, \quad \text{where} \quad k = \frac{\Gamma^3(1/3)}{2\pi\sqrt{3}},$$

then $g = \wp'$ where $\wp$ is the Weierstrass function with periods $\sqrt{3}$ and $\sqrt{3}e^{2\pi i/3}$. The Riemannian metric ρ_0 in the proof of the Theorem corresponds to the length element $|(g^{-1})'(w)|\,|dw|$.

The function

$$f_1(z) = g\left(\frac{\sqrt{3}}{2\pi i}\log z\right),$$

is evidently meromorphic in $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ and has three critical values. Simple calculation shows that it satisfies

$$A(r, f_1) = \frac{\sqrt{3}}{\pi}\log r + O(1), \quad r \to \infty.$$

Now we modify f_1 to obtain a function meromorphic in $\mathbf{C}$. Consider the integral

$$I(z) = \int_0^z \frac{d\zeta}{\zeta^{2/3}(1-\zeta)^{1/3}},$$

Using the positive value of the cubic root, and integrating over [0, 1] we obtain

$$I(1) = 2\pi/\sqrt{3}.$$

Using this, one can easily verify that

$$f_0(z) = g\left(\frac{\sqrt{3}}{2\pi i}I(z)\right)$$

is meromorphic in the plane. This function f_0 has three critical values, no asymptotic values, and satisfies (3), as a branch of $I(z)$ near infinity has the same asymptotic behavior as the logarithm.

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