In ordinary geometry a surface is conceived as a locus of points; in Lie's geometry it appears as the totality of all the spheres having contact with the surface. This gives a threefold infinity of spheres, or a complex of spheres,

$$F(a, b, c, d, e, r) = 0.$$

But this, of course, is not a *general* complex; for not every complex will be such as to touch a surface. It has been shown that the condition that must be fulfilled by a complex of spheres, if all its spheres are to touch a surface, is the following:

$$\left(\frac{\partial F}{\partial b}\right)^2 + \left(\frac{\partial F}{\partial c}\right)^2 + \left(\frac{\partial F}{\partial d}\right)^2 - \left(\frac{\partial F}{\partial r}\right)^2 - \frac{\partial F}{\partial a}\frac{\partial F}{\partial e} = 0.$$

To give at least one illustration of the further development of this interesting theory, I will mention that among the infinite number of spheres touching the surface at any point there are two having stationary contact with the surface; they are called the *principal spheres*. The lines of curvature of the surface can then be defined as curves along which the principal spheres touch the surface in two successive points.

Plücker's line-geometry can be studied by the same two methods just mentioned. In this geometry let $p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23}$ be the usual six homogeneous co-ordinates, where $p_{ik} = -p_{ki}$. Then we have the identity

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0,$$

and we take as group the ∞^{15} linear substitutions transforming this equation into itself. This group corresponds to the totality of collineations and reciprocations, *i.e.* to the projective group. The reason for this lies in the fact that the polar equation

$$p_{12}p_{34}' + p_{13}p_{42}' + p_{14}p_{23}' + p_{34}p_{12}' + p_{42}p_{13}' + p_{23}p_{14}' = 0$$

expresses the intersection of the two lines p, p'.

Now Lie has instituted a comparison of the highest interest between the line-geometry of Plücker and his own sphere-geometry. In each of these geometries there occur six homogeneous co-ordinates connected by a homogeneous equation of the second degree. The discriminant of each equation is different from zero. It follows that we can pass from either of these geometries to the other by linear substitutions. Thus, to transform

$$p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$$

into

$$b^2+c^2+d^2-r^2-ae=0,$$

it is sufficient to assume, say,

$$p_{12}=b+ic, \quad p_{13}=d+r, \quad p_{14}=-a,$$

$$p_{34}=b-ic, \quad p_{42}=d-r, \quad p_{23}=e.$$

It follows from the linear character of the substitutions that the polar equations are likewise transformed into each other. Thus we have the remarkable result that *two spheres that touch correspond to two lines that intersect.*

It is worthy of notice that the equations of transformation involve the imaginary unit i; and the law of inertia of quadratic forms shows at once that this introduction of the imaginary cannot be avoided, but is essential.

To illustrate the value of this transformation of line-geometry into sphere-geometry, and *vice versa,* let us consider three linear equations,

$$F_1=0, \; F_2=0, \; F_3=0,$$

the variables being either line co-ordinates or sphere co-ordinates. In the former case the three equations represent a *set of lines; i.e.* one of the two sets of straight lines of a hyperboloid of one sheet. It is well known that each line of either set intersects all the lines of the other. Transforming to sphere-geometry, we obtain a *set of spheres* corresponding to each set of lines; and every sphere of either set must touch every sphere of the other set. This gives a configuration well known in geometry from other investigations; viz. all these spheres envelop a surface known as Dupin's cyclide. We have thus found a noteworthy correlation between the hyperboloid of one sheet and Dupin's cyclide.

Perhaps the most striking example of the fruitfulness of this work of Lie's is his discovery that by means of this transformation the lines of curvature of a surface are transformed into asymptotic lines of the transformed surface, and *vice versa.* This appears by taking the definition given above for the lines of curvature and translating it word for word into the language of line-geometry. Two problems in the infinitesimal geometry of surfaces, that had long been regarded as entirely distinct, are thus shown to be really identical. This must certainly be regarded as one of the most elegant contributions to differential geometry made in recent times.

Lecture III.: SOPHUS LIE.

(August 30, 1893.)

The distinction between analytic and algebraic functions, so important in pure analysis, enters also into the treatment of geometry.

Analytic functions are those that can be represented by a power series, convergent within a certain region bounded by the so-called circle of convergence. Outside of this region the analytic function is not regarded as given *a priori;* its continuation into wider regions remains a matter of special investigation and may give very different results, according to the particular case considered.

On the other hand, an *algebraic* function, $w = \text{Alg.}(z)$, is supposed to be known for the whole complex plane, having a finite number of values for every value of z.

Similarly, in geometry, we may confine our attention to a limited portion of an analytic curve or surface, as, for instance, in constructing the tangent, investigating the curvature, etc.; or we may have to consider the whole extent of algebraic curves and surfaces in space.

Almost the whole of the applications of the differential and integral calculus to geometry belongs to the former branch of geometry; and as this is what we are mainly concerned with in the present lecture, we need not restrict ourselves to algebraic functions, but may use the more general analytic functions confining ourselves always to limited portions of space. I thought it advisable to state this here once for all, since here in America the consideration of algebraic curves has perhaps been too predominant.

The possibility of introducing new elements of space has been pointed out in the preceding lecture. To-day we shall use again a new space-element, consisting of an infinitesimal portion of a surface (or rather of its tangent plane) with a definite point in it. This is called, though not very properly, a *surface-element* (*Flächenelement*), and may perhaps be likened to an infinitesimal fish-scale. From a more abstract point of view it may be defined as simply the combination of a plane with a point in it.

As the equation of a plane passing through a point (x, y, z) can be written in the form

$$z' - z = p(x' - x) + q(y' - y),$$

x', y', z' being the current co-ordinates, we have x, y, z, p, q as the co-ordinates of our surface-element, so that space becomes a fivefold manifoldness. If homogeneous co-ordinates be used, the point (x_1, x_2, x_3, x_4) and the plane (u_1, u_2, u_3, u_4) passing through it are connected by the condition

$$x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4 = 0,$$

expressing their united position; and the number of independent constants is $3 + 3 - 1 = 5$, as before.

Let us now see how ordinary geometry appears in this representation. A point, being the locus of all surface-elements passing through it, is represented as a manifoldness of two dimensions, let us say for shortness, an M_2. A curve is represented by the totality of all those surface-elements that have their point on the curve and their plane passing through the tangent; these elements form again an M_2. Finally, a surface is given by those surface-elements that have their point on the surface and their plane coincident with the tangent plane of the surface; they, too, form an M_2.

Moreover, all these M_2's have an important property in common: any two consecutive surface-elements belonging to the same point, curve, or surface always satisfy the condition

$$dz - pdx - qdy = 0,$$

which is a simple case of a Pfaffian relation; and conversely, if two surface-elements satisfy this condition, they belong to the same point, curve, or surface, as the case may be.

Thus we have the highly interesting result that in the geometry of surface-elements points as well as curves and surfaces are brought under one head, being all represented by twofold manifoldnesses having the property just explained. This definition is the more important as there are no other M_2's having the same property.

We now proceed to consider the very general kind of transformations called by Lie *contact-transformations*. They are transformations that change our element (x, y, z, p, q) into (x', y', z', p', q') by such substitutions

$$x' = \phi(x, y, z, p, q), \quad y' = \psi(x, y, z, p, q), \quad z' = \cdots, \quad p' = \cdots, \quad q' = \cdots,$$

as will transform into itself the linear differential equation

$$dz - pdx - qdy = 0.$$

The geometrical meaning of the transformation is evidently that any M_2 having the given property is changed into an M_2 having the same property. Thus, for instance, a surface is transformed generally into a surface, or in special cases into a point or a curve. Moreover, let us consider two manifoldnesses M_2 having a contact, *i.e.* having a surface-element in common; these M_2's are changed by the transformation into two other M_2's having also a contact. From this characteristic the name given by Lie to the transformation will be understood.

Contact-transformations are so important, and occur so frequently, that particular cases attracted the attention of geometers long ago, though not under this name and from this point of view, *i.e.* not as contact-transformations, so that the true insight into their nature could not be obtained.

Numerous examples of contact-transformations are given in my (lithographed) lectures on *Höhere Geometrie*, delivered during the winter-semester of 1892–93. Thus, an example in two dimensions is found in the problem of wheel-gearing. The outline of the tooth of one wheel being given, it is here required to find the outline of the tooth of the other wheel, as I explained to you in my lecture at the Chicago Exhibition, with the aid of the models in the German university exhibit.

Another example is found in the theory of perturbations in astronomy; Lagrange's method of variation of parameters as applied to the problem of three bodies is equivalent to a contact-transformation in a higher space.

The group of ∞^{15} substitutions considered yesterday in line-geometry is also a group of contact-transformations, both the collineations and reciprocations, having this character. The reciprocations give the first well-known instance of the transformation of a point into a plane (*i.e.* a surface), and a curve into a developable (*i.e.* also a surface). These transformations of curves will here be considered as transforming the *elements* of the points or curves into the *elements* of the surface.

Finally, we have examples of contact-transformations, not only in the transformations of spheres discussed in the last lecture, but even in the general transition from the line-geometry of Plücker to the sphere-geometry of Lie. Let us consider this last case somewhat more in detail.

First of all, two lines that intersect have, of course, a surface-element in common; and as the two corresponding spheres must also have a surface-element in common, they will be in contact, as is actually the case for our transformation. It will be of interest to consider more closely the correlation between the surface-elements of a line and those of a sphere, although it is given by imaginary formulæ. Take, for instance, the totality of the surface-elements belonging to a circle on one of the spheres; we may call this a *circular set* of elements. In line-geometry there corresponds the set of surface-elements along a generating line of a skew surface; and so on. The theorem regarding the transformation of the curves of curvature into asymptotic lines becomes now self-evident. Instead of the curve of curvature of a surface we have here to consider the corresponding elements of the surface which we may call a *curvature set.* Similarly, an asymptotic line is replaced by the elements of the surface along this line; to this the name *osculating set* may be given. The correspondence between the two sets is brought out immediately by considering that two consecutive elements of a curvature set belong to the same sphere, while two consecutive elements of an osculating set belong to the same straight line.

One of the most important applications of contact-transformations is found in the theory of partial differential equations; I shall here confine myself to partial differential equations of the first order. From our new point of view, this theory assumes a much higher degree of perspicuity, and the true meaning of the terms "solution," "general solution," "complete solution," "singular solution," introduced by Lagrange and Monge, is brought out with much greater clearness.

Let us consider the partial differential equation of the first order

$$f(x, y, z, p, q) = 0.$$