On the hyperbolic metric of the complement of a rectangular lattice

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Abstract

The density of the hyperbolic metric on the complement of a rectangular lattice is investigated. The question is related to conformal mapping of symmetric circular quadrilaterals with all zero angles.

MSC: 30C20.

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0. Introduction. A famous theorem of Landau says that there exists $L > 0$ with the property that for all functions f analytic in the unit disc, such that $|f'(0)| = 1$, the image contains a disc of radius $L - \epsilon$, for every $\epsilon > 0$.

The largest constant L for which this is true is called the Landau constant, and its exact value is unknown. It is natural to conjecture that the extremal functions are universal coverings of complements to lattices in the plane.

For lattices with the property that the largest disc in the complement has radius 1, one can consider the problem of maximizing $|f'(0)|$ over all analytic functions mapping the unit disc to the complement of the lattice. Albert Baernstein II and J. Vinson [2] found that the hexagonal lattice and the universal covering with the property that $f(0)$ is a center of the complement give a strict local maximum in this problem.

This paper originated from an attempt of the author to prove a similar result for the class of rectangular lattices. A natural conjecture is that the square lattice must be extremal, and that for the extremal function $f(0)$ is

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a center of the complement. This is still unproved. However, the author feels that even the computation of the hyperbolic metric at the center of the complement of a rectangular lattice is an interesting problem. We will see that it is related to conformal mapping of certain symmetric circular quadrilaterals, whose all angles are zero, the problem which attracted attention of mathematicians and physicists during the whole XX century [3, 4, 7, 8, 9, 10].

We consider a rectangular lattice $\{2m\omega + 2n\omega'\}$ where $\omega = (\ln K)/2$, $K > 1$, and $\omega' = \pi i$. Let f be a universal cover of the complement of this lattice by the unit disc such that $f(0) = \omega + \omega'$ is a center. We are interested in the quantity $A(\ln K) = |f'(0)|$ as a function of $\ln K$.

In our use of the standard notation of the theory of elliptic functions we follow [5] (see also [1]): $\tau = \omega'/\omega$, $\quad h = e^{\pi i \tau}$; $\theta_j(\zeta)$ is the j-th theta-function, $\theta_j = \theta_j(0)$, and Jacobi's Modular Function is κ^2.

1. Denote

$$k = \frac{\ln K}{2\pi}, \quad a(k) = A(2\pi k) = A(\ln K) = |f'(0)|.$$

We may assume that f maps a circular quadrilateral Q (having zero angles, inscribed in the unit circle, symmetric with respect to reflections in coordinate axes) onto a fundamental rectangle R of the lattice, such that $f(0)$ is the center of R. Then a simple symmetry and rescaling argument gives the functional equation

$$a(k^{-1}) = k^{-1}a(k), \quad k > 0.$$

It is easy to see that in the limit when $k \to 0$, f maps the unit disc onto the strip $0 < |\Im u| < 2\pi$, and we obtain

$$a(0) = 4.$$

Together with the functional equation this implies

$$a(k) \sim 4k, \quad k \to \infty.$$

Differentiating the functional equation we obtain

$$a'(1) = \frac{1}{2}a(1).$$

In the next two sections we find $a(1)$ and $a'(0)$. In section 4, we derive an "explicit expression" for $a(k)$ for all k, and in section 5 discuss the computational aspects.

Finding the density of hyperbolic metric in the complement of a rectangular lattice is equivalent to finding a conformal map from a rectangle onto a hyperbolic quadrilateral with zero angles. This is a classical problem investigated by Hilbert [4] and Klein [8], and in modern times in [7]. The mapping satisfies a Schwarz differential equation related to a Lamé equation. The problem of finding this mapping for a given circular quadrilateral (not necessarily inscribed in a circle) requires determination of the so-called accessory parameter which is a solution of a transcendental equation involving Hill's determinants. See [3, 9, 10] for results in this direction. The only paper I know where the accessory parameter was actually computed is [7] but it does not contain a rigorous justification of the algorithm.[1] So we describe in sections 4 and 5 a convergent algorithm for the special case considered here.

2. Finding $a(1)$**.** We use the upper half-plane rather than the unit disc. Let T be the triangle in the upper half-plane with zero angles and vertices $0, 1, \infty$. Quadrilateral Q_1 is the union of T with its reflection about its left vertical side. Now the "center" is at the point $\tau = i$, and we are looking for $|f_1'(i)|$, where

$$f_1(\tau) = f(\zeta),$$

$\tau(\zeta)$ is a map prom the unit disc onto the upper halfplane, $\tau(0) = i$. Our function f_1 maps Q_1 onto a square R_1 of side 2π. We split it into a composition of two functions.

$$f_1 = g \circ \kappa^2,$$

where κ^2 is the Modular Function of Jacobi. (In [1] this function has double notation, sometimes λ, sometimes κ^2). κ^2 maps T onto the upper half-plane sending $(\infty, 0, 1)$ to $(0, 1, \infty)$. Thus $\kappa^2(i) = 1/2$.

Our second component is a Schwarz-Christoffel map

$$g(w) = C \int_0^w \zeta^{-3/4}(1-\zeta)^{-3/4} d\zeta,$$

[1] The authors say on p. 217: "It should be emphasized that our remarks about the implicit equations are purely heuristic and that the actual computation proceeded, as it were, fortuitously without any a priori justification.

with

$$C = \frac{2\pi\sqrt{2}}{B(1/4, 1/4)} \approx 1.981\,,$$

where B is Euler's Beta-function. This g maps the upper half-plane onto the isosceles right triangle with legs 2π, and $g(1/2)$ is the middle of the hypotenuse. We have

$$g'(1/2) = C \cdot 2^{3/2} = 3.887\,.$$

Using the notation from [1, 5], we have

$$\kappa^2 = (\theta_2/\theta_3)^4, \quad [5, \text{II},4, \S5],$$

$$\begin{aligned} \theta_2 &= 2h^{1/4}(1 + h^2 + h^6 + h^{12} + \ldots), \\ \theta_3 &= 1 + 2h + 2h^4 + 2h^9 + \ldots, \qquad \text{(see [5, II,2, §6])}, \end{aligned}$$

where

$$h = \exp(\pi i \tau).$$

Thus

$$\frac{d}{d\tau}\kappa^2 = 4\theta_2^3\theta_3^{-5}(\theta_2'\theta_3 - \theta_2\theta_3')\frac{dh}{d\tau}.$$

and $d\tau/d\zeta = 2$ at $z = 0$.

For $\zeta = 0$, $\tau = i$, $h = e^{-\pi}$, Matlab gives

$$|f'(0)| = 2|(\kappa^2)'(i)|g'(1/2) \approx 7.416.$$

3. Finding $a'(0)$. Let us make a preliminary map of the unit disc onto the horizontal strip $|\Im w| < 2\pi$, so that our quadrilateral Q is mapped onto the quadrilateral Q_2 bounded by two vertical segments of length 2π each, and two half-circles, orthogonal to the boundary of the strip, and zero corresponds to the center of Q_2, which is also at $w = 0$. We have

$$\left|\frac{dw}{d\zeta}\right| = 4 \quad \text{at the point} \quad \zeta = 0.$$

Suppose that Q_2 has width $\epsilon > 0$ which is small. Then f_2 maps Q_2 into a rectangle R_2, bounded by two vertical segments of length 2π and two horizontal segments, and also having small width. It is enough to estimate the width of this R_2 because

$$f_2'(0) \approx (\text{width of}\, R_2)/\epsilon.$$

Let us rescale both Q_2 and R_2 to the width 1, denoting the rescaled quadrilateral by Q_2' and R_2', and let G be the conformal map between them. Now we are interested in the ratio of the lengths of these quadrilaterals, which are large. If we cut Q_2' and R_2' by horizontal segments in the middle, the lower half of Q_2' will be conformally equivalent to the lower half of R_2' (as curvilinear quadrilaterals). Let us compare the restriction of G to these halfs with the conformal map F of the triangle T (see section 1) onto a vertical halfstrip with vertices $0, 1, \infty$ (and right angles at 0 and 1). Such map with the vertex correspondence $(0, 1, \infty) \to (0, 1, \infty)$ is given by

$$F(\tau) = \frac{1}{\pi} \arccos \frac{2 - \kappa^2(\tau)}{\kappa^2(\tau)}.$$

Using the explicit expression for κ^2 above and putting $\tau = iy$, we obtain

$$F(iy)/i = y - \frac{1}{\pi} \ln 4 + o(1), \quad y \to +\infty.$$

This means that R_2' is shorter than Q_2' by

$$\frac{2}{\pi} \ln 4 + o(1).$$

(That $G(iy) = F(iy) + o(1), y \to \infty$ is quite evident, and this is easy to justify by using extremal length or some other argument). Passing back to the original R_2 and Q_2 we obtain that the width of R_2 is

$$\epsilon + \frac{\epsilon^2}{\pi^2} \ln 4 + o(\epsilon^2).$$

This means that $a = |f'(0)| = 4|f_2'(0)| = 4 + 4\epsilon\pi^{-2} \ln 4 + o(\epsilon)$, that is

$$a'(0) = \frac{4}{\pi^2} \ln 4 \approx 0.5618.$$

4. Representation of $|f'(0)| = a(k)$ for arbitrary k.

Let $\wp$ be the Weierstrass function of our rectangular lattice (with real period 2ω and pure imaginary period $2\omega'$). We set

$$P(\zeta) = \frac{1}{4}(\wp(\zeta + \omega + \omega') - e_2).$$

Then P is real on both real and imaginary axis, in fact it maps the rectangle $R_0 = \{0 < \Re u < \omega, 0 < \Im u < \omega'/i\}$ (one quarter of the fundamental rectangle) onto the lower half-plane (see Fig. 1) P is holomorphic in the closure of R_0, except at one point $\omega + \omega'$ where it has a pole of second order.

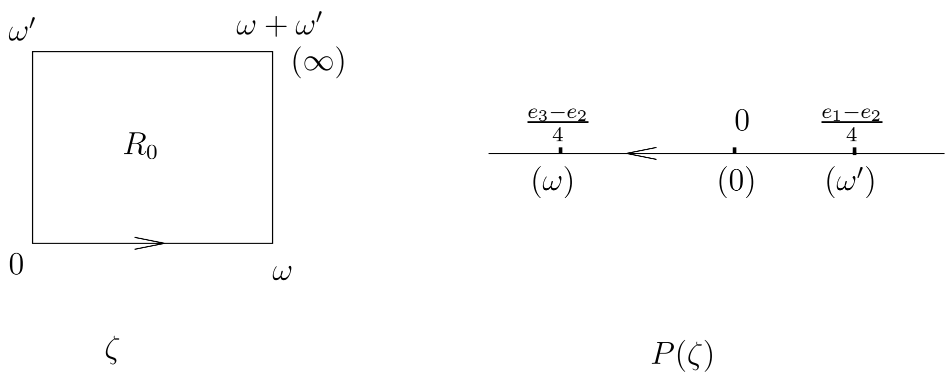


Figure 1. Conformal mapping by P.

Consider the differential equation

$$\frac{d^2 w}{d\zeta^2} + P(\zeta) w = \lambda w, \tag{1}$$

where $\lambda = \lambda(\omega, \omega')$ is a real parameter to be specified later.

Let λ^- be the largest eigenvalue of (1) with boundary conditions $w'(0) = w'(\omega) = 0$. Taking into account that $P(\zeta) < 0$ for $\zeta \in [0, \omega]$ (see Fig. 1), we conclude that

$$\lambda^- \leq 0. \tag{2}$$

Putting $\zeta = it$ in (1) we obtain

$$\frac{d^2 w}{dt^2} = (P(it) - \lambda) w. \tag{3}$$

Let λ^+ be the smallest eigenvalue of (3) with the boundary conditions $w'(0) = w'(\omega'/i) = 0$. Using $P(\zeta) > 0$ for $\zeta \in (0, \omega']$ (see Fig. 1), we obtain

$$\lambda^+ \geq 0. \tag{4}$$

Let c and s be two solutions of (1) defined by

$$\left(\begin{array}{cc} c(0) & s(0) \\ c'(0) & s'(0) \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).$$

Then $c(\sigma), s(\sigma)$ are real for $\sigma \in [0, \omega]$, while $c(it)$ is real and $s(it)$ is pure imaginary for $t \in [0, \omega']$. We introduce four real quantities:

$$m = (s(\omega)/c(\omega) + s'(\omega)/c'(\omega))/2, \tag{5}$$

$$r = (s(\omega)/c(\omega) - s'(\omega)/c'(\omega))/2, \tag{6}$$

$$m_1 = -i(s(\omega')/c(\omega') + s'(\omega')/c'(\omega'))/2, \tag{7}$$

$$r_1 = -i(s(\omega')/c(\omega') - s'(\omega')/c'(\omega'))/2. \tag{8}$$

Proposition. *There is a unique $\lambda \in [\lambda^-, \lambda^+]$ such that the condition*

$$\frac{m_1^2}{\sqrt{m_1^2 + m^2}} = r_1 \tag{9}$$

is satisfied. Then we have

$$a(k) = \frac{m m_1}{\sqrt{m^2 + m_1^2}}. \tag{10}$$

Proof of the Proposition. Consider the Schwarz differential equation associated with (1):

$$\frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = 2(P - \lambda). \tag{11}$$

It is well known (and easy to verify) that every solution of (11) is a ratio of two linearly independent solutions of (1). So we put $F = s/c$.

4.1. Then F is locally univalent, real on the real axis, and pure imaginary on the imaginary axis.

4.2. We claim that for $\lambda \in [\lambda^-, \lambda^+]$,

$$c(\zeta) \neq 0 \quad \text{when} \quad \zeta \in [0, \omega] \cup [0, \omega']. \tag{12}$$

This follows from the Sturm Comparison Theorem.

Thus F is holomorphic on $[0, \omega] \cup [0, \omega']$.

4.3. By the Symmetry principle,

$$F(\overline{\zeta}) = \overline{F(\zeta)} \quad \text{and} \quad F(-\overline{\zeta}) = -\overline{F(\zeta)}. \tag{13}$$

It is well-known that F maps the right vertical side L' and the top horizontal side L of the rectangle R_0 onto some arcs of circles on the Riemann sphere. We recall a simple proof of this. Consider, for example the right vertical side L'. As the right hand side of (11) has period 2ω, and all solutions of (11) are fractional linear transformations of each other, we conclude that $F(\zeta - 2\omega) = \lambda \circ F(\zeta)$. Using $L' - 2\omega = -\overline{L'}$ and (13), we obtain

$$-\overline{F(L')} = \lambda \circ F(L').$$

Thus $F(L')$ is fixed by an anticonformal involution, so it is an arc of a circle. Similar argument applies to $F(L)$.

4.4. It is known that the arcs $F(L)$ and $F(L')$ have exactly one common point z_0 and they are tangent at this point. We recall how this is proved. The coefficient P in (1) has one pole in the closure of R_0, namely at the point $\omega + \omega'$, and the Laurent series at this pole has the form

$$P(\omega + \omega' + \zeta) = \frac{1}{4\zeta^2} + \dots .$$

It follows that (1) has two linearly independent solutions near $\omega + \omega'$ of the form

$$\phi(\omega + \omega' + \zeta) = \zeta^{1/2} + \dots \quad \text{and} \quad \psi(\omega + \omega' + \zeta) = \zeta^{1/2} \ln \zeta + \dots$$

(see, for example, [6]). Thus F (which is a fractional-linear transformation of ϕ/ψ) has a limit z_0 as $\zeta \to \omega + \omega'$ from inside of R_0, and the angle between

$F(L)$ and $F(L')$ at z_0 is zero. As $F(L)$ and $F(L')$ are arcs of circles, z_0 is the only point of intersection of these circles.

4.5. We have shown in 4.1-4.4 that F maps the boundary ∂R_0 locally univalently onto a Jordan curve (consisting of a segment of the real axis, a segment of the imaginary axis, and two arcs of circles tangent to each other, one perpendicular to the real axis another to the imaginary axis).

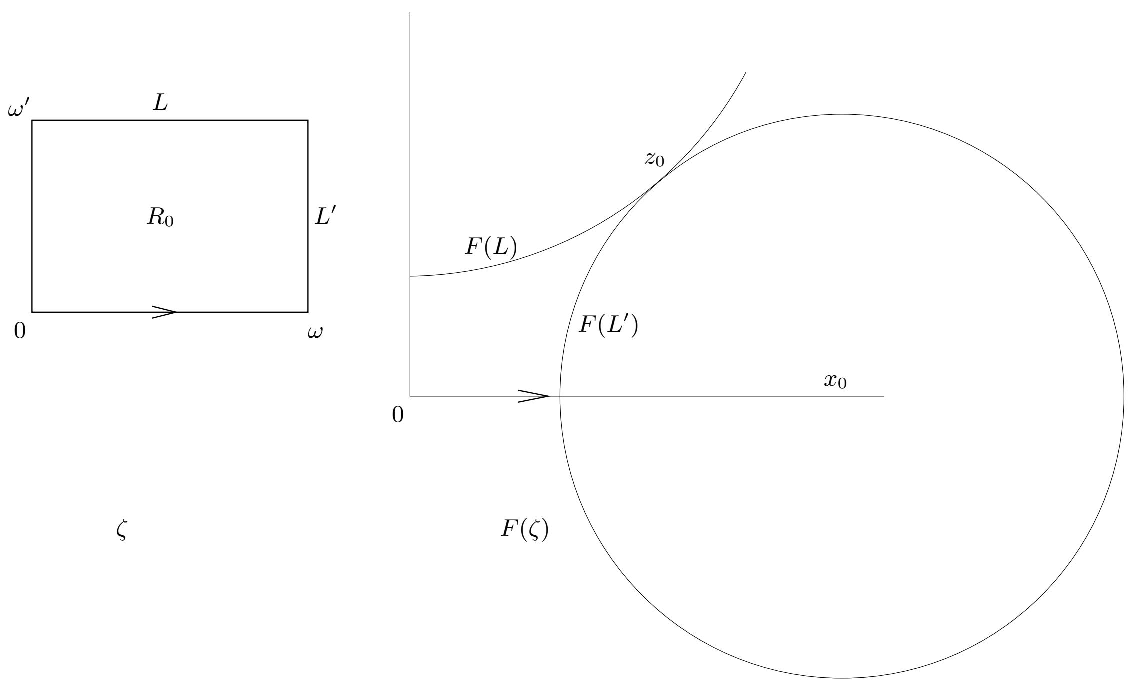


Figure 2. Conformal mapping by F.

We claim that right circle has center at the point m and radius r, while the top circle has center im_1 and radius r_1 (the definitions of these four numbers are given in (5).

Let us verify the claim for the right circle. It is clear that the matrix

$$\begin{pmatrix} s(\omega) & s'(\omega) \\ c(\omega) & c'(\omega) \end{pmatrix}$$

is real. Introducing the pair of solutions (ϕ, ψ) of the equation (3), normalized by

$$\begin{pmatrix} \phi(\omega) & (d\phi/dt)(\omega) \\ \psi(\omega) & (d\psi/dt)(\omega) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where} \quad \zeta = it,$$

we see that ϕ and ψ are real on the vertical line $\{\omega + it : t \in \mathbf{R}\}$ (as solutions of real differential equation (3) with real initial conditions), and conclude that

$$\begin{aligned} s &= s(\omega)\phi + is'(\omega)\psi, \\ c &= c(\omega)\phi + ic'(\omega)\psi, \end{aligned}$$

and thus the image $F(L')$ belongs to the circle

$$\left\{ \frac{s(\omega) + s'(\omega)ix}{c(\omega) + c'(\omega)ix} : x \in \mathbf{R} \right\},$$

whose center is at the point m and radius is r.

The verification for the other circle is similar.

Now we choose our parameter λ in such a way that the common tangent line to the two circles passes through the origin. Elementary geometry shows that this happens iff the condition (9) is satisfied, see Figure 3.

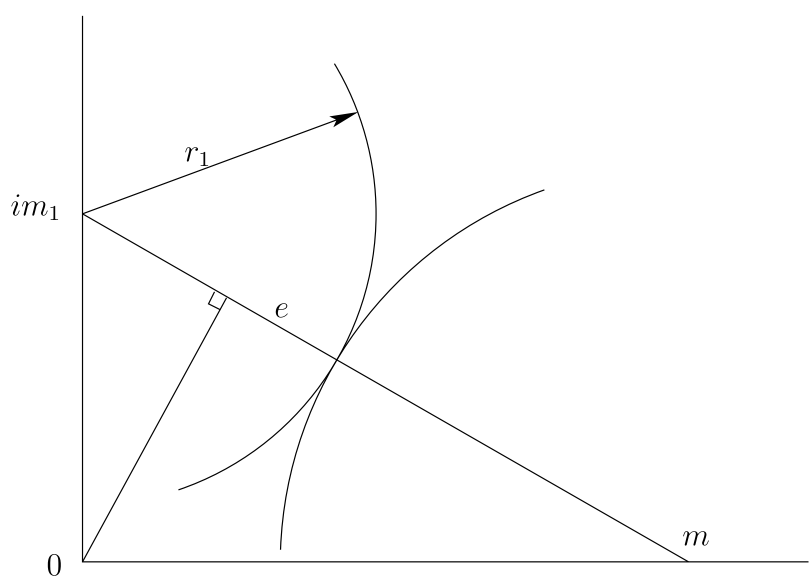


Figure 3. Choosing the accessory parameter.

When $\lambda \to \lambda^-$, the right circle becomes a vertical line, and when $\lambda \to \lambda^+$ the top circle becomes a horizontal line. It follows that the quantity $e = m_1^2/\sqrt{m^2+m_1^2} - r_1$ changes sign when λ changes on the interval $[\lambda^-, \lambda^+]$. It follows that there exists λ such that condition (9) is satisfied. In fact this value of λ is unique but we do not use this additional information.

For this value of λ, F maps our rectangle R_0 onto a quadrilateral Q_0 bounded by two straight segments $F([0,\omega]) \subset \mathbf{R}$, $F([0,\omega']) \subset i\mathbf{R}$ and two arcs of circle tangent at the point z_0. Reflecting this quadrilateral Q_0 three times with respect to the coordinate axes we obtain a circular quadrilateral symmetric with respect to the coordinate axes, inscribed in the circle $\{z : |z| = |z_0|\}$ and sides perpendicular to this circle. Thus F is the inverse to the universal covering of the complement of our lattice by the disc $\{z : |z| < |z_0|\}$, $F(0) = 0$, $F'(0) = 1$, and it remains to verify that the radius $|z_0|$ of this disc is given by the formula (10). This is clear from Figure 3.

5. Remarks on computation. Our method of computation is based on the previous section. We are solving the Lamé equation (1). We represent P in terms of theta-functions or in terms of elliptic functions of Jacobi (see [5, 1]):

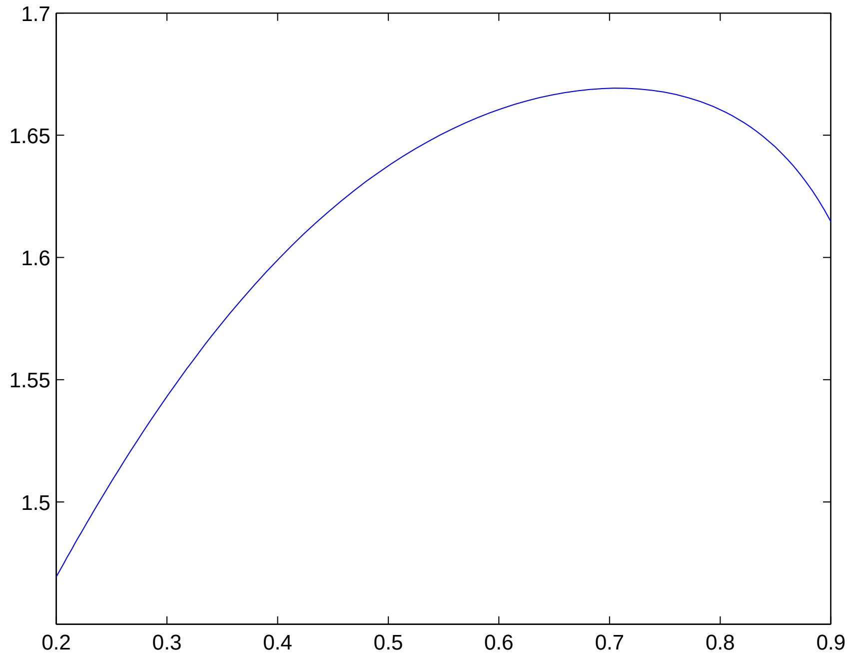
$$P(\zeta) = -\frac{{\theta_1'}^2\, \theta_1^2(\zeta/(2\omega))}{16\omega^2\, \theta_3^2\, \theta_3^2(\zeta/(2\omega))}$$

where either product or series representations can be used for theta functions, they converge very well. Numerical solving of the equations (1) and (3) on the real line involves only computations with real numbers.

As Matlab does not have standard routines for theta functions, we can use an expression of $\wp$ in terms of Jacobi elliptic functions. Matlab uses AGM to compute them which is probably as effective as theta functions.

To find the accessory parameter λ we use a "shooting method", dissecting the interval $[\lambda^-, \lambda^+]$ dyadically.

The result is that $a(1) \approx 1.4163$ and the graph of $|f'(0)|$ for the rectangular lattice with sides $2\omega, 2\omega'$, $4\omega^2 + 4{\omega'}^2 = 1$, as a function of 2ω is this:



We see a maximum at the point $1/\sqrt{2}$ and this maximum equals 1.6693.

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