

Tangencies between holomorphic maps and holomorphic laminations

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Abstract

We prove that the set of leaves of a holomorphic lamination of codimension one that are tangent to a germ of a holomorphic map is discrete.

Let F be a *holomorphic lamination* of codimension one in an open set V in a complex Banach space B . In this paper, this means that $V = W \times \mathbf{C}$, where W is a neighborhood of the origin in some Banach space, and the leaves L_λ of the lamination are disjoint graphs of holomorphic functions $\mathbf{w} \mapsto f(\lambda, \mathbf{w})$, $W \rightarrow \mathbf{C}$. For holomorphic functions in a Banach space we refer to [5]. Here λ is a parameter and we assume that the dependence of f on λ is continuous. A natural choice of this parameter is such that $\lambda = f(\lambda, 0)$, in which case the continuity with respect to λ follows from the so-called λ -lemma of Mane-Sullivan-Sad and Lyubich, see, for example [5]. With this choice of the parameter, our definition of a lamination coincides with that of a holomorphic motion of \mathbf{C} parametrized by W .

Let $\gamma : U \rightarrow V$ be a holomorphic map, $U \subset \mathbf{C}^n$. We say that γ is *tangent* to the lamination at a point $\mathbf{z}_0 \in U$ if the image of the derivative $\gamma'(\mathbf{z}_0)$ is contained in the tangent space $T_L(\gamma(\mathbf{z}_0))$, where L is the leaf passing through $\gamma(\mathbf{z}_0)$. A leaf for which this holds is called a *tangent leaf* to γ .

Theorem. *Let K be a compact subset of U . Then the set of leaves tangent to γ at the points of K is finite.*

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For the case of holomorphic curves ($n = 1$) this result is contained in [1, Lemma 9.1] where it is credited to Douady. Artur Avila, in a conversation with the authors, proposed to extend this result to arbitrary holomorphic maps. According to Avila, this generalization has several applications to holomorphic dynamics [1].

Proof. We assume without loss of generality that

$$f(0, \mathbf{w}) \equiv 0,$$

and that L_0 is tangent to γ at $\mathbf{z}_0 = 0$.

We have to show that other tangent leaves cannot accumulate to L_0 . Suppose the contrary, that is suppose that there is a sequence $\lambda_k \rightarrow 0$ such that L_{λ_k} are tangent to γ , and let L_{λ_k} be the graphs of the functions $f_k(\mathbf{w}) = f(\lambda_k, \mathbf{w})$. We may assume that tangency points $\mathbf{z}_k \rightarrow 0$.

We make several preliminary reductions.

1. Let

$$\gamma(\mathbf{z}) = (\phi(\mathbf{z}), \psi(\mathbf{z})) \in W \times \mathbf{C}.$$

Consider the new lamination in $U \times \mathbf{C}$ whose leaves are the graphs of $f^*(\lambda, \mathbf{z}) = f(\lambda, \phi(\mathbf{z}))$ and the new map $\gamma^*(\mathbf{z}) = (\mathbf{z}, \psi(\mathbf{z}))$. Then γ^* is tangent to a leaf L^* if and only if γ is tangent to L . This reduces our problem to the case that W is an open set in \mathbf{C}^n and the map γ is a graph of a function ψ of the same variable as the functions f_k . From now on we assume that $U = W$ and $\gamma(\mathbf{w}) = (\mathbf{w}, \psi(\mathbf{w}))$.

2. Now we reduce the problem to the case that ψ is a monomial. For this we use the desingularization theorem of Hironaka [4, 2, 3, 6].

Let X be a complex analytic manifold, and ψ an analytic function on X . Then there exists a complex analytic manifold M and a proper surjective map $\pi : M \rightarrow X$ such that the restriction of π onto the complement of the π -preimage of the set $\{\psi = 0, \psi' = 0\}$ is injective and for each point $\mathbf{z}_0 \in \pi^{-1}(\{\psi = 0\})$ there is a local coordinate system with the origin at \mathbf{z}_0 such that $\psi \circ \pi$ is a monomial $z_1^{m_1} \dots z_n^{m_n}$. We choose such M corresponding to $X = W$.

Let $Y = W \times \mathbf{C}$, and let $S \subset Y$ be the set of points (\mathbf{w}, t) with $t \neq 0$ such that the graph of $t = \psi(\mathbf{w})$ is tangent to the lamination. In our proof by contradiction, we assume that the origin belongs to the closure of S . Let $N = M \times \mathbf{C}$, and let $\rho : N \rightarrow Y$ be the map defined by $\rho(\mathbf{z}, t) = (\pi(\mathbf{z}), t)$.

Then $\rho^{-1}(F)$ is the lamination whose leaves are the components of the ρ -preimages of the leaves of F , and the set $T = \rho^{-1}(S)$ has a limit point $(\mathbf{z}_0, 0)$ with $\pi(\mathbf{z}_0) = 0$ since π is proper. Also the set T is exactly the set of those points in N where the graph $t = \psi \circ \pi(\mathbf{z})$ is tangent to the lamination $\rho^{-1}(F)$ since ρ is injective in a neighborhood of each point of T . (Any point (\mathbf{z}, t) where ρ is not injective satisfies $\psi(\pi(\mathbf{z})) = 0$ while at every point of T we have $\psi(\pi(\mathbf{z})) \neq 0$.) This reduces our problem to the case that ψ is a monomial.

3. We may assume now that $W = \{\mathbf{z} : |\mathbf{z}| < 2\}$. So we are in the following situation.

$$\psi(z) = z_1^{m_1} \dots z_n^{m_n},$$

and $\{f_k\}$ is a family of holomorphic functions on W with disjoint graphs, $f_k(\mathbf{z}) \neq 0$ for $\mathbf{z} \in W$, and $f_k \rightarrow 0$ as $k \rightarrow \infty$ uniformly on W . Moreover, for some sequence $\mathbf{z}_k \rightarrow 0$ we have

$$f_k(\mathbf{z}_k) = \prod_{j=1}^n z_{j,k}^{m_j}, \quad (1)$$

$$\text{grad } f_k(\mathbf{z}_k) = (m_1 z_{1,k}^{m_1-1} z_{2,k}^{m_2} \dots, m_2 z_1^{m_1} z_{2,k}^{m_2-1} \dots, \dots) \quad (2)$$

assuming zero values for the components with $m_j = 0$. We may assume that $|f_k| \leq 1$ in W . Setting $f_k = \exp g_k$ we obtain that $\Re g_k \leq 0$. Now we put

$$h_k(\mathbf{z}) = g_k(\mathbf{z} + \mathbf{z}_k) - g_k(\mathbf{z}_k).$$

Then the h_k are defined in the unit ball and satisfy

$$\Re h_k(\mathbf{z}) \leq \sum_{j:m_j>0} m_j \log |z_{j,k}|^{-1}.$$

From this we conclude that

$$\left| \frac{\partial h_k}{\partial z_j}(0) \right| \leq 2 \sum_{j:m_j>0} m_j \log |z_{j,k}|^{-1}. \quad (3)$$

This follows from the

Lemma. *Let h be an analytic function in the unit disc, $h(0) = 0$, and $\Re h \leq A$, where $A > 0$. Then $|h'(0)| \leq 2A$.*

Proof. The function

$$\phi(z) = \frac{z}{2Az - z}$$

maps the half-plane $\{\Re z < A\}$ onto the unit disc, $\phi(0) = 0$, and $\phi'(0) = 1/(2A)$. Now the statement follows from the Schwarz lemma applied to $\phi \circ h$ and the chain rule.

On the other hand, (1) and (2) imply that, for $m_j > 0$,

$$\left| \frac{\partial h_k}{\partial z_j}(0) \right| = \frac{m_j}{|z_{j,k}|}. \quad (4)$$

Assume without loss of generality that $m_1 > 0$ and

$$|z_{1,k}| = \min_{j:m_j>0} |z_{j,k}|.$$

Then the RHS of (3) is at most

$$\text{const} \log |z_{1,k}|^{-1},$$

while the RHS of (4) for $j = 1$ is

$$\frac{m_1}{|z_{1,k}|}.$$

As $|z_{1,k}| \rightarrow 0$, we obtain a contradiction which proves our theorem.

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