Properties of Laplace transform

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Let f be a function satisfying $|f(t)| \leq Ce^{At}$ for some positive C, A. We denote by $F = L[f]$ the Laplace transform:

$$F(s) = \int_0^\infty e^{-st} f(t) dt.$$

Then F is defined on the ray $s > A$ and has the follolwing properties (we denote by G the Laplace transform of some other similar function g, and define

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases}$$

the Heaviside function).

1. Linearity: $L[c_1 f + c_2 g] = c_1 F + c_2 G$.
2. $L[e^{ct} f(t)] = F(s-c)$, and for $a > 0$

$$L[u_a(t) f(t-a)] = e^{-as} F(s).$$

3. For $a > 0$, $L[f(at)] = a^{-1} F(a^{-1} s)$.
4. $L[f'] = sF(s) - f(0)$, and by applying this several times

$$L[f^{(n)}] = s^n F(s) - s^{n-1} f(0) - \ldots - f^{n-1}(0).$$

5. $L\left[\int_0^t f(\tau) d\tau\right] = s^{-1} F(s)$.
6. $L[tf(t)] = -F'(s)$.
7. $L[t^{-1} f(t)] = \int_s^\infty F(x) dx$.

8. $L[f \star g] = FG$.

There are several inversion formulas for Laplace transform, but we only need to know that the function f is uniquely determined by its Laplace transform, which follows from the existence of an inversion formula. One of these inversion formulas is

$$f(t) = \lim_{n \to \infty} \frac{(-1)^{n-1}}{(n-1)!} \left(\frac{n}{t}\right)^n F^{(n-1)}\left(\frac{n}{t}\right).$$