# Lecture 3.31. Jordan form

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Today we finally address deficient matrices, that is those which are non-diagonalizable, which do not have enough eigenvectors to span the space.

A typical example is the following $m \times m$ matrix

$$J_m(\lambda_1) = \begin{pmatrix} \lambda_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_1 & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_1 & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_1 \end{pmatrix}.$$

We have $\lambda_1$ on the main diagonal, 1 on the diagonal above the main, all other entries are 0. Such a matrix is called a *Jordan block* of size $m$ with eigenvalue $\lambda_1$. Its characteristic polynomial is $(\lambda_1 - \lambda)^m$, so the only eigenvalue is $\lambda_1$, and the eigenspace corresponding to this eigenvalue is 1-dimensional. It is spanned by an eigenvector $(1, 0, \dots, 0)^T$. Check all this!

Look how this matrix maps the vectors of the standard basis:

$$Ae_1 = \lambda_1 e_1,\ Ae_2 = \lambda e_2 + e_1,\ Ae_3 = \lambda_1 e_3 + e_2,$$

and so on. So $e_1$ is an eigenvector, and the rest of $e_j$ are called *generalized eigenvectors*.

**Definition.** Let $A$ be a square matrix, and $\lambda$ be an eigenvalue and $\mathbf{v} \neq 0$ an eigenvector, so that $A\mathbf{v} = \lambda\mathbf{v}$. Then the first generalized eigenvector attached to $\mathbf{v}$ is a solution $\mathbf{v}_1$ of the equation

$$A\mathbf{v}_1 = \lambda\mathbf{v}_1 + \mathbf{v}.$$

The second generalized eigenvector $\mathbf{v}_2$ attached to $\mathbf{v}$ is a solution of

$$A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1.$$

and so on. Generalized eigenvectors form a chain $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2, \ldots, \mathbf{v}_k$ such that

$$A\mathbf{v}_{j+1} = \lambda\mathbf{v}_{j+1} + \mathbf{v}_j.$$

**Jordan's Theorem**. *For every linear operator $L$ in a (complex) finite-dimensional space there is a basis consisting of eigenvectors and generalized eigenvectors.*

It is called the Jordan basis for $L$.

I recall that two matrices $A$ and $B$ are called *similar* if there is a non-singular matrix $C$ such that

$$A = CBC^{-1}.$$

Geometrically, this means they they represent the same linear operator, and encode it with respect to two different bases.

The matrix of the operator with respect to its Jordan basis has a *Jordan form* which consists of diagonal blocks, each block is a Jordan block.

**Corollary.** *Every square matrix is similar to its Jordan form. Two matrices are similar if and only if they have the same Jordan form (up to permutation of Jordan blocks).*

If $A$ is a matrix, and $J$ is its Jordan form, then

$$A = BJB^{-1},$$

where $B$ is the matrix whose columns are eigenvectors and generalized eigenvectors.

**Examples.** Here are all possible Jordan forms for $n = 2$:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}.$$

The first of these is diagonalizable, it has two Jordan blocks of size 1. The second has one Jordan block of of size 2.

Next we list all possible Jordan forms for $n = 3$:

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

Jordan forms obtained by permutation of the blocks are considered the same. First matrix has 3 blocks of size 1, second has 2 blocks of sizes 2 and 1, and the third matrix has one block of size 3.

First matrix is diagonalizable ($\lambda_1, \lambda_2, \lambda_3$ are not necessarily distinct!): it has three linearly independent eigenvectors.

The second has two linearly independent eigenvectors, one with eigenvalue $\lambda_1$ and one with eigenvalue $\lambda_2$ (again $\lambda_1$ and $\lambda_2$) are not necessarily distinct!). It also has one generalized eigenvector attached to the true eigenvector corresponding to $\lambda_1$.

The third matrix has only one eigenvector (up to proportionality) and to it two generalized eigenvectors are attached.

In general, suppose we have some eigenvalue $\lambda$. The dimension of the eigenspace corresponding to $\lambda$ is the number of Jordan blocks with this $\lambda$ on the main diagonal. To each of these eigenvector, some number of generalized eigenvectors can be attached. The sum of sizes of all Jordan blocks with eigenvalue $\lambda$ is the multiplicity of $\lambda$ as a root of the characteristic equation.

So, for example, this matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

has characteristic equation $\lambda^5 - 32 = 0$. The only eigenvalue is $\lambda = 2$. The eigenspace corresponding to this eigenvalue has dimension 2. So we have two linearly independent eigenvectors, they are in fact $e_1$ and $e_4$. In addition we have generalized eigenvectors: to $e_1$ correspond two of them: first $e_2$ and second $e_3$. To the eigenvector $e_4$ corresponds a generalized eigenvector $e_5$.

To find the Jordan form and the Jordan basis for some matrix, you do the following:

a) find eigenvalues.

b) for each eigenvalue, find a basis of the eigenspace. If the sum of the dimensions of eigenspaces is $n$, the matrix is diagonalizable, and your eigenvectors make a basis of the whole space.

c) if not, try to find generalized eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$ by solving $(A - \lambda I)\mathbf{v}_1 = \mathbf{v}$, for an eigenvector $\mathbf{v}$, then, if not enough, $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$, and so on. You need only one solution of each of these equations.

These generalized eigenvectors will be attached to those $\lambda$ for which the number of true linearly independent eigenvectors is less than the multiplicity of $\lambda$ as the root of the characteristic equation.

d) Jordan's Theorem guarantees you that eventually you will find sufficient number of generalized eigenvectors, so that they, together with true eigenvectors make a basis of the whole space. This is the Jordan basis.

**Example 1.** (Probl. 40, p. 305) *Which pairs of these matrices are always similar (for all $a, b, c, d$) and which are not (for some $a, b, c, d$)?*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} b & a \\ d & c \end{pmatrix}, \quad C = \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \quad D = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Answer: $A$ and $D$ are similar. $B$ and $C$ are similar. All other pairs may not be similar for some $a, b, c, d$.

Explanation: $A$ and $D$ have the same characteristic polynomial. Also $B$ and $C$ have the same characteristic polynomial. But these two polynomials are different, and it is easy to find $a, b, c, d$ for which they are numerically different.

Now $A$ is similar to $D$ because

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Multiplication on

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is self-inverse, from the left interchanges rows and multiplication on the same from the right interchanges columns. Similarly $B$ is similar to $C$.

**Example 2.** Find the Jordan form and the Jordan basis for the following matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

*Solution.* The characteristic polynomial is $(2-\lambda)^3$, so the only eigenvalue is 2. By the standard procedure we find two linearly independent eigenvectors: $(0,0,1)^T$ and $(1,2,0)^T$. None of them has a generalized eigenvector, so one has to try some linear combination of them, for example $(1,2,1)^T$. To this eigenvector, there is a generalized eigenvector $(0,1,0)^T$. So the Jordan form and a Jordan basis can be taken as

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The main use of the Jordan form is for solving differential and difference equations with deficient matrices. One can compute the powers and exponential of the Jordan block (see the book, p. 300-301). The $k$-th power is upper triangular and has the following form:

on the main diagonal stand $\lambda^k$.

on the diagonal above the main stand $k\lambda^{k-1}$,

on the next diagonal stand $\frac{k(k-1)}{2}\lambda^{k-2}$,

and so on. On the diagonal number $m \le k$ above the main diagonal stand

$$\frac{k!}{m!(k-m)!}\lambda^{k-m},$$

the terms of the binomial formula.

The exponential $e^{Jt}$ of the Jordan block is also upper triangular:

on the main diagonal: $e^{\lambda t}$,

on the next above: $te^{\lambda t}$,

then on the next: $(t^2/2)e^{\lambda t}$, and so on.

on the $m$-th diagonal above the main: $(t^m/m!)e^{\lambda t}$.

**Exercise.** Prove these facts.

In practice, one does the following. Consider a differential equation

$$\mathbf{x}' = A\mathbf{x}.$$

Let $\lambda$ be an eigenvalue of $A$ and $\mathbf{v}$ an eigenvector. To it corresponds a solution

$$\mathbf{x}_0(t) = e^{\lambda t}\mathbf{v}.$$

let $\mathbf{v}_1$ be the first generalized eigenvector. To it corresponds the solution

$$\mathbf{x}_1(t) = e^{\lambda t}(t\mathbf{v} + \mathbf{v}_1).$$

indeed

$$\mathbf{x}_1' = \lambda t e^{\lambda t}\mathbf{v} + e^{\lambda t}(\mathbf{v} + \lambda \mathbf{v}_1).$$

Indeed, plugging this to the equation and dividing by $e^{\lambda t}$ we obtain

$$t\mathbf{v} + (\mathbf{v} + \lambda \mathbf{v}_1) = (tA\mathbf{v} + A\mathbf{v}_1),$$

and this is true since $A\mathbf{v} = \lambda \mathbf{v}$ and $Av_1 = \lambda v_1 + v$. Similarly, if there is a second generalized eigenvector, the form of the third solution is

$$\mathbf{x}_2(t) = (t^2/2)e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2.$$

And in general, for the $(m+1)$st generalized eigenvector,

$$\mathbf{x}_m(t) = (t^m/m!)e^{\lambda t}\mathbf{v} + (t^{m-1}/(m-1)!)e^{\lambda t}\mathbf{v}_1 + \ldots + \mathbf{v}_m.$$

**Exercise.** Verify this.

**Effect of the Jordan form on stability.** When $\operatorname{Re}\lambda < 0$, $e^{\lambda t} \to 0$ as $t \to \infty$. This does not change if we multiply the exponent on any power of $t$. Therefore when all eigenvalues have negative real part, the system $\mathbf{x}' = A\mathbf{x}$ has stable equilibrium at 0, no matter whether $A$ is diagonalizable or not.

However, when $\operatorname{Re}\lambda = 0$, then $e^{\lambda t}$ is bounded, but it oscillates. Multiplication on $t$ gives an unbounded function. So when all eigenvalues have non-positive real part, this is not sufficient for stability. If those eigenvalues with $\operatorname{Re}\lambda = 0$ are not deficient (have as many linearly independent eigenvectors as their multiplicities as roots of the characteristic equation) then the system is stable. However if there is at least one eigenvalue with $\operatorname{Re}\lambda = 0$ which is deficient (has fewer eigenvectors than its multiplicity) then the system is not stable: each generalized eigenvector will give an unbounded solution.

**Remark.** If a matrix is known only approximately, for example, if its entries are results of some measurements, and computation indicates that the characteristic equation has a multiple root, one cannot conclude whether this root is indeed multiple, and if it is, whether the root is deficient or not (whether there are as many eigenvectors as its multiplicity, or less). Many computer systems have a command called Jordan, which is supposed to find the Jordan form, but one has to be very careful with them: it is easy to deceive these programs, even with $2 \times 2$ matrices!