Lecture 4.16. Extremal Principles

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Let A and M be symmetric matrices, with M *positive definite*. Then the corresponding quadratic forms can be simultaneously reduced to linear combinations of squares, which means that there exists a non-singular matrix C such that

$$C^T MC = I, \quad C^T AC = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \tag{1}$$

where λ_j are *generalized eigenvalues* solving the generalized characteristic equation $\det(A - \lambda M) = 0$.

All eigenvalues are real, and we label them in the non-decreasing order, each eigenvalue is repeated according to its multiplicity so that:

$$\lambda_1 \leq \lambda_2 \ldots, \leq \lambda_n.$$

The purpose of the following is to give formulas of the generalized eigenvalues which do not depend on any choice of coordinates.

Let us introduce the function, called the *Rayleigh ratio*, which is defined for $x \neq 0$:

$$R(x) = \frac{x^T Ax}{x^T Mx}, \quad x \neq 0.$$

Our goal is to study its extremal properties. Notice that R is homogeneous, $R(kx) = R(x)$ for any real k. It follows that it has a maximum and a minimum.

Consider the transformation $x = Cy$, where C is the non-singular matrix from (1). We have $y = C^{-1}x$, so the coordinates y_k of y are $y_k = u_k^T x$, where u_k^T is the k-th row of C^{-1}. We also denote by v_k the k-th column of C, so that $u_i^T v_j = \delta_{i,j}$.

We have in view of (1) with $x = Cy$:

$$R(x) = \frac{y^T \Lambda y}{y^2} = \frac{\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2}{y_1^2 + \ldots + y_n^2}. \tag{2}$$

From this representation, it is immediately evident that

$$\min_x R(x) = \lambda_1 \quad \text{and} \quad \max_x R(x) = \lambda_n.$$

So we obtained representations of λ_1 and λ_n which are completely independent of any coordinates.

This representation permits easy estimates of eigenvalues.

Example let $A = (a_{i,j})$ be a symmetric matrix whose smallest eigenvalue is λ_1 and the largest eigenvalue λ_n. Computing the Rayleigh ratio on the vector $\mathbf{x} = (1, 1, \ldots, 1)^T$ we obtain

$$\mathbf{x}^T A \mathbf{x} = \sum_{i,j} a_{i,j}, \quad \mathbf{x}^T \mathbf{x} = n, \quad R(\mathbf{x}) = \frac{1}{n} \sum_{i,j} a_{i,j},$$

so the Rayleigh ratio is simply the arithmetic average of all entries of A, and we obtain

$$\lambda_1 \leq \frac{1}{n} \sum_{i,j} a_{i,j} \leq \lambda_n.$$

Using other vectors, we obtain infinitely many inequalities which give explicit estimates of λ_1 from above and λ_n from below.

To obtain a similar representation for the rest of λ_j, we consider a minimization problem with restrictions. First of all

$$\min_{u_1^T x = 0} R(x) = \min_{y_1 = 0} R(x) = \min \frac{\lambda_2 y_2^2 + \ldots + \lambda_n y_n^2}{y_2^2 + \ldots + y_n^2} = \lambda_2. \tag{3}$$

This is not very useful, because the knowledge of u_1 is required. So take *any vector* a and consider the restriction $a^T x = 0$. We will show that minimum with this restriction is between λ_1 and λ_2. That it is at least λ_1 is clear because λ_1 is the unrestricted minimum. To show that it is at most λ_2, let us choose a vector $x = c_1 v_1 + c_2 v_2 \neq 0$, which satisfies the restriction $a^T x = 0$. The restriction gives *one* homogeneous linear equation on two unknowns c_1, c_2, namely

$$c_1 a^T v_1 + c_2 a^T v_2 = 0$$

therefore such a non-zero solution (c_1, c_2) exists. For this vector x,

$$R(x) = \frac{\lambda_1 c_1^2 + \lambda_2 c_2^2}{c_1^2 + c_2^2} \leq \lambda_2,$$

and x satisfies the restriction. So the restricted minimum is between λ_1 and λ_2. Combined with (3) this can be stated as:

$$\max_a \min_{a^T x = 0} R(x) = \lambda_2.$$

This gives a formula for λ_2 as a solution of a *maximin problem.* Completely similar reasoning gives the following:

Maximin Principle.

$$\lambda_k = \max_{a_1, \ldots, a_{k-1}} \left(\min_{a_1^T x = 0, \ldots, a_{k-1}^T x = 0} R(x) \right).$$

Proof. When $a_1 = u_1, \ldots, a_{k-1} = u_{k-1}$ we have

$$\min_{u_1^T x = 0, \ldots, u_{k-1}^T x = 0} R(x) = \min \frac{\lambda_k y_k^2 + \ldots + \lambda_n y_n^2}{y_k^2 + \ldots + y_n^2} = \lambda_k.$$

Now consider any restriction

$$a_1^T x = \ldots a_{k-1}^T x = 0,$$

and choose a non-zero vector

$$x = y_1 v_1 + \ldots + y_k v_k$$

which satisfies the restriction. This is possible, becauses the restrictions are $k - 1$ equations and we have k unknowns, so there is always a non-zero solution. For this vector

$$R(x) = \frac{\lambda_1 y_1^2 + \ldots + \lambda_k y_k^2}{y_1^2 + \ldots + y_k^2} \leq \lambda_k.$$

This completes the proof.

Similarly we could begin with maximizing $R(x)$ instead of minimizing. Then completely similar arguments give the

Minimax Principle

$$\lambda_k = \min_{a_1,\dots,a_{n-k}} \left(\max_{a_1^T x=0,\dots,a_{n-k}^T x=0} R(x) \right).$$

Applications.
1. Let us say that $A \geq B$ if $x^T Ax \geq x^T Bx$ for all $x \neq 0$. This is equivalent to saying that $A - B$ is positive semi-definite. Indeed

$$x^T Ax - x^T Bx = x^T (A - B)x.$$

Now consider four matrices A, M, A', M' all symmetric, and M, M' positive definite. Let λ_j be the generalized eigenvalues solving $\det(A - \lambda M) = 0$ and λ'_j the generalized eigenvalues solving $\det(A' - \lambda' M') = 0$.

Theorem 1. *If $A \geq A'$ and $M \leq M'$ then $\lambda_j \geq \lambda'_j$, for all j.*

Indeed, we have $R(x) \geq R'(x)$ for the Rayleigh ratios, so all maxima and minima involving R are at least those involving R'.

This has a nice and useful physical interpretation. Recall the equation of small oscillation of a mechanical system. It is a second order differential equation

$$My'' + Ky = 0,$$

where M (mass) and K (stiffness) are symmetric matrices, and $M > 0$. So we obtain the following principle:

Increasing stiffness and/or decreasing mass of the system results in increasing all frequencies of proper oscillations.

2. Suppose that the matrix A' is obtained from a symmetric matrix A by deleting some columns and rows with the same numbers. For example, A' can be a NW submatrix of A. How are their eigenvalues related? Suppose that A is of size $n \times n$ and A' is of size $m \times m$, $m < n$. Let $\lambda'_1 \leq \lambda'_2 \leq \ldots \leq \lambda'_m$ be eigenvalues of A' and $\lambda_1 \leq \ldots \leq \lambda_n$ be eigenvalues of A.

Theorem 2. *We have*

$$\lambda_k \leq \lambda'_k \leq \lambda_{k+n-m}, \quad 1 \leq k \leq m. \tag{4}$$

In particular, when $n - m = 1$ we obtain

$$\lambda_1 \le \lambda'_1 \le \lambda_2 \le \lambda'_2 \le \ldots \le \lambda'_{n-1} \le \lambda_n.$$

This is called the *interlacing property*: the eigenvalues of the reduced matrix interlace with those of the original one.

Example. What can we say about eigenvalues of the following $n \times n$ matrix:

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 2 \\ 0 & 0 & \ldots & 0 & 3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 2 & \ldots & n-1 & n \end{pmatrix} ?$$

Let us cross out the last column and the last row. If the eigenvalues of the original matrix are $\lambda_1, \ldots, \lambda_n$ and the eigenvaluesr of the new matrix are $\lambda'_1, \ldots, \lambda'_{n-1}$, then they must be interlaced:

$$\lambda_1 \le \lambda'_1 \le \lambda_2 \le \ldots \le \lambda_{n-1} \le \lambda'_{n-1} \le \lambda_n.$$

But $\lambda'_j = 0$ for all j, so we conclude that the original matrix has $\lambda_1 \le 0 \le \lambda_n$ and zero is an eigenvalue of multiplicity at least $n - 2$.

Let us test this for $n = 4$. Expanding the determinant along the first row, we obtain

$$\begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -\lambda & 0 & 2 \\ 0 & 0 & -\lambda & 3 \\ 1 & 2 & 3 & 4-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 2 \\ 0 & -\lambda & 3 \\ 2 & 3 & 4-\lambda \end{vmatrix} - \begin{vmatrix} 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \\ 1 & 2 & 3 \end{vmatrix}$$
$$= \lambda^4 - 4\lambda^3 - 14\lambda.$$

So the eigenvalues are $\lambda_{4,1} = 2 \pm \sqrt{18}$, of different signs, and $\lambda_2 = \lambda_3 = 0$, as predicted.

Now let us cross out

Proof of Theorem 2. To prove $\lambda'_k \ge \lambda_k$ we use Maximin Principle. λ'_k can be written as maximum over $k-1$ restraints of minima $R(\mathbf{x})$ under these $k-1$ restrains and the restraints $x_{m+1} = \ldots = x_n = 0$. Removing the last $n - m$ restraints decreases the minimum of $R(\mathbf{x})$, and maximum over restraints of minima of $R(\mathbf{x})$ under $k - 1$ restraints is λ_k.

To prove $\lambda'_k \leq \lambda_{k+n-m}$ we use Maximin Principle again. For $x \in \mathbf{R}^n$ we denote by $\mathbf{x}' \in \mathbf{R}^m$ the vector consisting of the first m coordinates of $\mathbf{x}$. Then

$$\lambda'_k = \max_{\text{restrictions}} \left(\min_{k-1 \text{ restrictions}} R'(x') \right)$$

But minimum of $R'(\mathbf{x})$ under $k-1$ restrictions equals to the minimum of $R(\mathbf{x})$ with the same restrictions plus $n-m$ restrictions of the form $x_j = 0 \; m+1 \leq j \leq n$. So the total number of restrictions is $k-1+n-m$. This does not exceed maximum of these minima over all possible $k-1+n-m$ restrictions and this is λ_{k+n-m}.

Instead of considering submatrices, we can impose arbitrary linear restrictions that is a restriction of the quadratic form on a subspace of dimension m. The result will be the same.

Example. Watch the following short movies on YouTube:

https://www.youtube.com/watch?v=LrAZq8uxB3c&t=26s

https://www.youtube.com/watch?v=i2QyehtQTFQ

What's going on? A railway worker walks along the train with a long hummer, and kicks the axles of railway carriages. His job is called "wheel tapper". He checks for the cracks in the following way: the sound produced by a cracked axle differs in pitch from the sound of a normal axle. Why does this happen, and in how exactly it differs? Consider a model of the solid as a set of moleculas connected by rods and/or springs. A crack means that some rods or springs are removed. The pitch of the sound is determined by the lowest frequency of the oscillating solid, that is the square root of the smallest eigenvalue. When we remove springs or rods, this smallest eigenvalues decreases. So the pitch of a cracked axle is lower than the pitch of an uncracked one.

I also recommend the following movie which demontrates the *modes* of oscullations. Modes are essentially the different eigenvectors, corresponding to different eigenvalues.

https://www.youtube.com/watch?time_continue=2&v=9N1aYy8Q9jo

3. Suppose that a form $\mathbf{x}^T A' \mathbf{x}$ is obtained from $\mathbf{x}^T A \mathbf{x}$ by adding r squares of linearly independent linear forms:

$$\mathbf{x}^T A' \mathbf{x} = x^T A \mathbf{x} + \sum_{j=1}^{r} (L_j(\mathbf{x}))^2. \tag{5}$$

How are the eigenvalues of A and A' related? Let λ_j be eigenvalues of A and λ'_j eigenvalues of A', both sequences are ordered so that they increase.

Theorem 3. *If the forms A and A' are related as in (5) then we have*

$$\lambda'_k \geq \lambda_k, \quad 1 \leq k \leq n$$

and

$$\lambda'_k \leq \lambda_{k+r}, \quad 1 \leq k \leq n - r.$$

Proof. The first inequality is clear from either Maximin or from Minimax Principle, because we have $R' \geq R$, so whatever maxima or minima one takes, this inequality will be preserved.

The second inequality is derived from the Maximin Principle. λ_{k+r} is the maximum over restrictions of minima of $R(\mathbf{x})$ with $k + r - 1$ restrictions. It will decrease if we take maximum not over all possible restrictions but fix r of them to be $L_j(\mathbf{x}) = 0$. But then this is the same as the minimum of $R'(\mathbf{x})$ with $k - 1$ restrictions and this is λ'_k.