Lecture 4.7. Bilinear and quadratic forms

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In this part of the course we will study *quadratic* functions on vector spaces. Vector spaces in this section are real by default.

Definitions. Let V be a real vector space. A function which assigns to every pair of vectors $\mathbf{x}, \mathbf{y}$ a real number $B(\mathbf{x}, \mathbf{y})$ is called a *bilinear form* if it is linear with respect to each vector:

$$B(c_1\mathbf{x}_1 + c_2\mathbf{x}_2, \mathbf{y}) = c_1 B(\mathbf{x}_1, \mathbf{y}) + c_2 B(\mathbf{x}_2, \mathbf{y}), \tag{1}$$

$$B(\mathbf{x}, c_1\mathbf{y}_1 + c_2\mathbf{y}_2) = c_1 B(\mathbf{x}, \mathbf{y}_1) + c_2 B(\mathbf{x}, \mathbf{y}_2). \tag{2}$$

A bilinear form is called *symmetric* if $B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y}$.

We recognize here the first two properties of the (real) dot product. $B(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$, where $(\mathbf{x}, \mathbf{y})$ is a dot product, is an example of a symmetric bilinear form. Here is another example, which is not symmetric:

$$B_1(\mathbf{x}, \mathbf{y}) = x_1 y_2 - x_2 y_1. \tag{3}$$

Actually this is the determinant of the 2×2 matric with columns $\mathbf{x}, \mathbf{y}$, so it is anti-symmetric (switches sign when the columns are exchanged). Here is an example of a symmetric bilinear form which is not a dot product:

$$B_2(\mathbf{x}, \mathbf{y}) = x_1 y_1 - x_2 y_2. \tag{4}$$

For a given basis in the space we can represent bilinear forms by matrices.

Let a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be given. Expand

$$\mathbf{x} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n, \quad \text{and} \quad \mathbf{y} = b_1\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n.$$

then by definition of a bilinear form we have

$$B(\mathbf{x},\mathbf{y}) = B\left(\sum_{j=1}^{n} a_j\mathbf{v}_j, \sum_{j=1}^{n} b_j\mathbf{v}_j\right) = \sum_{i,j} B(\mathbf{v}_i,\mathbf{v}_j)a_ib_j.$$

So we have an $n\times n$ matrix $A = (B(\mathbf{v}_i,\mathbf{v}_j))$ which is called the *Gram matrix* of the bilinear form, and

$$B(\mathbf{x},\mathbf{y}) = \mathbf{a}^T A\mathbf{b} = \sum_{i,j} a_i a_{i,j} b_j,$$

where $\mathbf{a}$ and $\mathbf{b}$ are column vectors of coordinates of $\mathbf{x}$ and $\mathbf{y}$.

In particular, if a standard basis is used, then the Gram matrix is $A = (B(e_i,e_j))$ and

$$B(\mathbf{x},\mathbf{y}) = \mathbf{x}^T A\mathbf{y}.$$

For example, to the forms B_1 and B_2 in (3) and (4) correspond matrices

$$A_1 = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \quad \text{and} \quad A_1 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right).$$

We will be mostly interested in *symmetric* bilinear forms. Their Gram matrices are *symmetric*.

To every symmetric bilinear form $B(\mathbf{x},\mathbf{y})$ corresponds a function of one vector $Q(x) = B(x,x)$. For example, if B is a dot product, then Q is the corresponding squared norm. This function Q is called the **quadratic form**, corresponding to the symmetric bilinear form B. For example, to the symmetric bilinear form B_2 above corresponds the quadratic form $Q(\mathbf{x}) = x_1^2 - x_2^2$.

It is a simple and and useful fact that the quadratic form completely determines the symmetric bilinear form from which it comes. In other words, we can recover $B(\mathbf{x},\mathbf{y})$ from the knowledge of Q only. This is done by the formula:

$$B(\mathbf{x},\mathbf{y}) = \frac{1}{4}\left(Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}-\mathbf{y})\right).$$

Exercise. Show that this is true if $Q(\mathbf{x}) = B(\mathbf{x},\mathbf{x})$, using the definiton (1), (2) and symmetry of B.

Every quadratic form on $\mathbf{R}^n$ can we written in the standard basis as

$$Q(\mathbf{x}) = \sum_{i,j} a_{i,j}x_ix_j = \mathbf{x}^T A\mathbf{x}, \quad \text{where} \quad A = (a_{i,j})$$

is a symmetric matrix. The correspondence between quadratic forms and symmetric matrices is one-to-one, when a basis is fixed.

So quadratic forms are simply polynomials in n variables, where each monomial has degree 2. (Degree of a monomial is the sum of its degrees with respect to all variables.) In general, the word "form" is a term for a homogeneous polynomial, that is a polynomial in which all monomials are of the same degree.

Examples:

$$Q(\mathbf{x}) = x_1^2 + x_1x_2 + 3x_2^2$$

is represented by the symmetric matrix

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 3 \end{pmatrix}.$$

$$x_1^2 + x_1x_2 + 5x_2^2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Notice where the $1/2$ comes from! ($x_1x_2 = (1/2)x_1x_2 + (1/2)x_2x_1$).

We are going to address two problems:

1. How to simplify a quadratic form by choosing an appropriate basis? (This is similar to Jordan theorem which shows how to simplify a linear operator by choosing an appropriate basis).

2. Which quadratic forms are positive, that is $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$. Solution of this problem will describe all possible dot products, since a dot product in a symmetric bilinear form for which the quadratic form has this positivity property.

To address problem 1, we investigate what happens to the matrix of a quadratic form when we change the basis. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be some basis in $\mathbf{R}^n$, and consider the matrix C whose columns are these basis vectors. Then C is non-singular, and every vector $\mathbf{x}$ can be written as

$$\mathbf{x} = C\mathbf{y},$$

where $\mathbf{y}$ is the column of coordinates of $\mathbf{x}$ with respect to the basis $\mathbf{v}_i$. Then for a quadratic form we have

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = (C\mathbf{y})^T AC\mathbf{y} = \mathbf{y}^T(C^T AC)\mathbf{y}.$$

So the matrix of the same quadratic form in the new basis is C^TAC.

(Notice the difference between quadratic forms and linear operators! If we change the basis, the matrix A of a linear operator changes to $C^{-1}AC$.)

This justifies the following

Definition. Two matrices A and B are called *congruent* if $A = C^TBC$ for some non-singular C.

One can say that congruent matrices represent the same quadratic form in different bases. (Like similar matrices represent the same linear operator in different bases).

Let

$$Q(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$$

be a quadratic form. So A is a symmetric matrix, and we can use the Spectral theorem for symmetric matrices to write

$$A = B\Lambda B^{-1} = B\Lambda B^T, \quad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

where λ_j are real. Here we used that B is orthogonal, so $B^{-1} = B^T$.

So we can write

$$Q(\mathbf{x}) = \mathbf{x}^TA\mathbf{x} = x^TB\Lambda B^T\mathbf{x} = y^T\Lambda\mathbf{y},$$

where $\mathbf{y} = B^T\mathbf{x}$. We can consider this as a change of coordinates, and in new coordinates $\mathbf{y}$, our form is represented by a diagonal matrix:

$$Q = \lambda_1y_1^2 + \lambda_2y_2^2 + \ldots + \lambda_ny_n^2.$$

So every quadratic form can be brought to such form by an *orthogonal* change of the basis. Such a new coordinate system $\mathbf{y}$ is traditionally called *a system of principal axes for* Q. We will later discuss the geometric meaning in detail.

Can one further simplify this representation? If all $\lambda_j > 0$ we can introduce a new coordinate system $\mathbf{z}$, where $z_j = y_j\sqrt{\lambda_j}$ and in this new coordinates our form will have the expression

$$Q = z_1^2 + \ldots + z_n^2.$$

If not all λ are positive, we can stil define $z_j = y_j\sqrt{|\lambda_j|}$, and $z_j = y_j$ when $\lambda_j = 0$, and obtain

$$Q = \pm z_1^2 \pm z_2^2 \pm \ldots \pm z_n^2. \tag{5}$$

In other words, there is always a coordinate system in which the expression of Q is just a signed sum of the squares of coordinates. Notice that some z_j can be absent in this sum! (They correspond to $\lambda_j = 0$.)

The relation between $\mathbf{z}$ and $\mathbf{y}$ is a non-singular transformation: it is represented by a diagonal matrix with positive entries.

So a quadratic form is characterized by its *signature*, which is triple of integers showing how many $+$ signs and $-$ signs and 0's are there in the representation (5).

It is convenient to write the signture as a sequence of $+, -$ and 0. So for example, the form $z_1^2 + z_3^2 - z_4^2$ in $\mathbf{R}^4$ has signature $(+, +, -, 0)$. The order of terms is not essential.

One can state this fact as follows:

Theorem. *Every real symmetric matrix A is congruent to a diagonal matrix with zeros, ones and minus ones on the main diagonal.*

The question arises: can a matrix be similar to two such diagonal matrices with different numbers of zeros, ones and minus ones. (Again we do not distinguish diagonal matrices obtained by permutation of diagonal entries). The answer is "no".

The Law of Inertia for quadratic forms. *Two diagonal matrices with different numbers of zeros ones and minus ones are not congruent.*

This means that every quadratic form has a well defined signature which is independent of its representation by a matrix. And every square symmetric matrix also has a signature, so that two matrices are congruant if and only if their signatures are the same.

To prove the law of inertia, we have to express somehow these three numbers in a geometric (basis-independent) way. This can be done in the following way.

Let Q be a quadratic form, and define

$$n_+ = \max\{\dim U : Q(\mathbf{x}) \geq 0 \quad \text{for all} \quad x \in U\}.$$

In words: n_+ is the maximum of dimensions of all subspaces such that the restriction of Q on U is non-negative.

This number n_+ depends only on the form Q, and is independent of the basis and matrix representation of Q. And it is easy to see that for a form represented as a signed sum of squares as in (5), this number is equal to the number of non-negative λ_j.

So similar matrices have the same numbers n_+. Similarly, we show that the number

$$n_- = \max\{\dim U : Q(\mathbf{x}) \leq 0 \quad \text{for all} \quad x \in U\}.$$

is invariant under congruence. This proves the Law of Inertia. Indeed, the number of zeros in the signature is $n_+ + n_- - n$, where n is the dimension of the space. Then the nuber of pluses is n_+ minus the number of zeros, and the number of minuses is n_- minus the number of zeros. So all three numbers are recovered from n_+, n_- and the dimension of the whole space n.

So all quadratic forms are distinguished by their sigunatures, and congruent forms have the same signature. Moreover, two forms are congruent if and only if they have the same signature.

To show that every quaratic form can be brought to the form (5) we used the Spectral theorem. And we know that finding eigenvalues is difficult. So the question arises whether we can find a change of the variable which brings the form to the form (5), and to find its signature *without finding the eigenvalues*.

This is indeed the case, and can be achieved by the following algorithm.

Algorithm of bringing a quadratic form to the signed sum of squares and determining its signature.

1. Suppose that Q has a pure square, for example, the term ax_1^2. Consider all terms which contain x_1 and "complete the square". We write all terms containing x_1 as

$$ax_1^2 + x_1 L(x_2, \ldots, x_n)$$

and transform it like this:

$$= a\left(x_1 + \frac{1}{2a}L(x_2, \ldots, x_a\right)^2 + -\frac{b^2}{4a}L^2(x_2, \ldots, x_n).$$

After that, x_1 will enter only to a square (of the term in big parentheses, and the rest will contain no x_1. Then the procedure can be repeated.

2. If Q has no term ax_j^2, but has a term of the form ax_1x_2, for example, then make the change of the variable

$$x_1 = u + v, \quad x_2 = u - v,$$

so that this term becomes $a(u^2 - v^2)$ and we can do step 1.

By performing repeatedly steps 1 and 2, we obtain a sum of the squares (with signs) at the end.

Examples.

1.

$$x_1^2+x_1x_2+3x_2^2=(x_1^2+x_1x_2+x_2^2/4)-x_2^2/4+3x_2^2=(x_1+x_2/2)^2+(11/4)x_2^2.$$

The signature is $(+,+)$.

2.

$$x_1x_2+x_2x_3=(u^2-v^2)+(u-v)x_3=u^2-v^2+ux_3-vx_3$$
$$=(u+x_3/2)^2-x_3^2/4-v^2-vx_3=(u+x_3/2)^2-(v+x_3/2)^2.$$

The signature is $(+,-,0)$.

The total number of squares that you obtain in the end is at most n (the dimension of the whole space).

This algorithm has a very nice application. Recall that using the Spectral theorem, we wrote a quadratic form $Q(\mathbf{x})=\mathbf{x}^TA\mathbf{x}$ as

$$Q(\mathbf{x})=\lambda_1y_1^2+\ldots+\lambda_ny_n^2,$$

where λ_j are eigenvalues of A.

On the other hand, we have a rational algorithm (using only arithmetic operations) for finding the signature. Thus we have a rational algorithm for determining how many eigenvalues of a symmetric matrix are positive, negative and zero! Without computing them.

For example, the second example above corresponds to the matrix

$$\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

We computed the signature, and this implies that the matrix has one positive, one negative and one zero eigenvalue. We did this without solving a cubic equation. And this works for any symmetric matrix.