

Zeros and coefficients

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Abstract

Asymptotic distribution of zeros of sequences of analytic functions is studied.

Let f_n be a sequence of analytic functions in a region D in the complex plane. The case of entire functions, $D = \mathbf{C}$ will be the most important. Suppose that there exist positive numbers $V_n \rightarrow \infty$, such that the subharmonic functions

$$\frac{1}{V_n} \log |f_n| \quad \text{are uniformly bounded from above} \quad (1)$$

on every compact subset of D , and for some $z_0 \in D$ the sequence

$$\frac{1}{V_n} \log |f_n(z_0)| \quad \text{is bounded from below.} \quad (2)$$

Under these conditions, one can select a subsequence such that for this subsequence

$$\frac{1}{V_n} \log |f_n| \rightarrow u, \quad (3)$$

where u is a subharmonic function in D , $u \neq -\infty$. This convergence can be understood in various senses, for example, quasi-everywhere, which means at every point except a set of points of zero logarithmic capacity, and also in D' (Schwartz distributions), or in L_{loc}^p , $1 \leq p < \infty$.

The corresponding Riesz measures converge weakly. The limit measure, which is $(2\pi)^{-1}\Delta u$, describes the limit distribution of zeros of f_n .

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For all these facts we refer to [9] or [10].

This setting is useful in many cases when one is interested in asymptotic distribution of zeros of sequences of analytic functions.

Let us mention several situations to which our results apply.

1. Let f_n be monic polynomials of degree n , and $V_n = n$. We denote by μ_n the so-called empirical measures, which are the Riesz measures of $(1/n) \log |f_n|$. The measure μ_n is discrete, with finitely many atoms, with an atom of mass m at every zero of f_n of multiplicity m . So the μ_n are probability measures in the plane. Therefore, one can always choose a subsequence for which $\mu_n \rightarrow \mu$, where μ is some measure in the plane. The sequence of potentials will converge:

$$\frac{1}{n} \log |f_n| \rightarrow u = \int_{|\zeta| \leq 1} \log |z - \zeta| d\mu(\zeta) + \int_{|\zeta| > 1} \log |1 - z/\zeta| d\mu(\zeta) + C.$$

2. Let f be an entire function, $f(0) \neq 0$, and

$$M(r, f) = \max\{|f(z)| : |z| = r\} \tag{4}$$

its maximum modulus. Suppose that for some sequence $r_n \rightarrow \infty$ we have

$$\log M(2r_n, f) = O(\log M(r_n, f)), \quad n \rightarrow \infty.$$

Then the functions $f_n(z) = f(r_n z)$ satisfy (1) and (2) with $V_n = \log M(r_n)$ and $z_0 = 0$.

3. The order of an entire function is defined by

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}.$$

An entire function of finite order is said to be of normal type if

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^\rho} < \infty.$$

If f is of finite order and normal type, $f(0) \neq 0$, and r_n is any sequence tending to infinity, then one can take $f_n(z) = f(r_n z)$, and $V_n = r_n^\rho$, and the properties (1) and (2) (with $z_0 = 0$) will hold.

Such sequences of subharmonic functions were applied to the asymptotic study of zeros in many papers. Of the early ones, we mention [4]. See also [1], [2], [3], [5], [11], [12], [13] and literature in these books and papers.

The purpose of this note is to state and prove two general facts about this convergence, and give several applications. There is no claim of novelty of the main results or their proofs, our only goal is to bring these general facts to the attention of a wider audience, as on our opinion they deserve.

The result of section 3 was found in the 1990s jointly with M. Sodin. This section is independent on section 1.

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1 Maximum modulus and coefficients

In this section the functions f_n are defined and analytic in a disc $D = \{z : |z| < R\}$ where $0 < R \leq \infty$, and we assume (1), (2) and (3).

Let $M(r, f_n)$ be the maximum modulus, and $B(r) = B(r, u) = \max_{|z|=r} u(z)$, then we have $\log M(r, f_n)/V_n \rightarrow B(r)$, $n \rightarrow \infty$. This convergence is uniform on every interval $[0, r_0]$ with $r_0 < R$. Let

$$\Phi(t) = B(e^t), \quad -\infty < t < \log R.$$

Suppose that

$$f_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k, \tag{5}$$

and let

$$\psi_n(x) = \sup_{\text{convex } \psi} \left\{ \psi : \psi \left(\frac{k}{V_n} \right) \leq -\frac{\log |a_{n,k}|}{V_n} \right\}.$$

Here the sup is taken over all convex functions ψ on the real line. To check that the class is not empty, we use the assumption that our functions f_n are analytic in some disc centered at 0 which implies that $\log |a_{n,k}| \leq C_n k$, so $\psi(x) = -C_n x$ is in the class.

It is convenient to consider all convex functions as defined on the whole real line and taking values in $(-\infty, +\infty]$. A convex function with finite values on a half-line or on an interval (closed, open, semi-open) can be always extended to the whole real line by setting it equal to $+\infty$ at every point outside this interval.

The set where a convex function is finite is always convex (but can be neither open nor closed). Then pointwise sup of any family of convex functions is convex, and the limit of decreasing sequence of convex functions is convex if finite at least at one point. Thus our functions ψ_n are convex.

The set where a convex function is discontinuous can consist of at most two points, the endpoints of the interval where it is finite. Lower semi-continuous regularization is still convex and differs from the original function at at most 2 points. We always assume our convex functions to be lower semi-continuous, making this regularization when necessary.

Theorem 1. *Suppose that (3) holds. Then $\psi_n \rightarrow \phi > -\infty$, and ϕ is the Legendre transform of Φ ,*

$$\phi(x) = \sup_t (tx - \Phi(t)).$$

Conversely, if $\liminf_{n \rightarrow \infty} \psi_n = \phi > -\infty$, then (1) holds and one can choose a subsequence for which (3) holds, and $\psi_n \rightarrow \phi$, and Φ and ϕ are related by the Legendre transform.

Proof. First we recall that Legendre's transform is defined for convex functions of the real line. The Legendre transform is always a convex, lower semi-continuous function. The transform is order-reversing: $f \geq g$ implies $L(f) \leq L(g)$. Finally, the Legendre transform is an involution, $L \circ L = \text{id}$.

We have by Cauchy inequality

$$|a_{n,k}|r^k \leq M(r, f_n),$$

Applying log, dividing by V_n , and writing $\log r = t$, we rewrite this as

$$\frac{-\log |a_{n,k}|}{V_n} \geq \sup_t \left(\frac{k}{V_n} t - \frac{1}{V_n} \log M(e^t, f_n) \right).$$

We set $x = k/V_n$, and as the RHS is convex in x , we obtain that

$$\psi_n(x) \geq \sup_t (xt - \log M(e^t, f_n)) / V_n.$$

We pass to the limit when $n \rightarrow \infty$ while x is fixed, and obtain

$$\phi(x) \geq \sup_t (xt - \Phi(t)). \tag{6}$$

Now we need an estimate in the opposite direction. Let us first give the estimate for the important special case when f_n are polynomials of degree n , and $V_n = n$. In this case our argument is very simple. We start with the trivial inequality

$$n \max_k |a_{n,k}| r^k \geq M(r, f_n). \quad (7)$$

Taking logs, and substituting $\log r = t$, we obtain

$$\max_k (kt + \log |a_{n,k}|) \geq \log M(e^t, f_n) - \log n.$$

Dividing by $V_n = n$ and passing to the limit as $n \rightarrow \infty$ with our usual arrangement that $k/n = x$, we obtain

$$\max_x (xt - \phi(x)) \geq \Phi(t). \quad (8)$$

Now, Legendre's transform is an involution, and order reversing. So our two inequalities imply the equality: ϕ and Φ are Legendre transforms of each other.

Returning to the general case, we need a substitute for (7). This is provided by rudimentary Wiman–Valiron theory, which we recall. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be a power series with radius of convergence $R \leq +\infty$. For every $r \in [0, R)$, we introduce the maximal term

$$m(r, f) = \max_k |a_k| r^k,$$

and the central index $\nu(r)$ as the largest integer for which

$$m(r, f) = |a_{\nu(r)}| r^{\nu(r)}.$$

For $0 < r < r_1 < R$, we have

$$\nu(r) \log \frac{r_1}{r} \leq \log m(r_1) - \log m(r). \quad (9)$$

For our functions f_n as in (5), notice the relation

$$\frac{1}{V_n} \log m(e^t, f_n) = \max_x (xt - \psi_n(x)) = L(\psi_n)(x). \quad (10)$$

Lemma 1. (Valiron). *For every r and r_1 such that*

$$0 \leq r < r_1 < R,$$

we have

$$M(r, f) \leq m(r_1, f) \left(\nu(r) + \frac{r_1}{r_1 - r} \right). \quad (11)$$

Proof.

$$\begin{aligned} M(r, f) &\leq \sum_{k=0}^{\infty} |a_k| r^k \\ &\leq m(r, f) \nu(r) + \sum_{k=\nu(r)+1}^{\infty} |a_k| r_1^k \left(\frac{r}{r_1} \right)^k \\ &\leq m(r_1, f) \nu(r) + m(r_1, f) \frac{r_1}{r_1 - r}. \end{aligned}$$

This is a sufficient substitute for (7) for our purpose.

Combining (9) and (11), we obtain

$$M(r, f_n) \leq m(r_1, f_n) \left(\frac{\log m(r_1, f_n) - \log m(r, f_n)}{\log r_1 - \log r} + \log r_1 - \log(r_1 - r) \right).$$

Taking logarithms, dividing by V_n and passing to the limit as $n \rightarrow \infty$, we take into account that

$$\lim_{n \rightarrow \infty} (\log m(r, f_n)) / V_n = L(\phi)(\log r),$$

and obtain that

$$\Phi(t) \leq L(\phi)(t'),$$

which holds for every $t < t'$. As both sides are continuous, we conclude that

$$\Phi \leq L(\phi)$$

which is (8). This completes the proof.

Theorem 1 gives an asymptotic connection between the maximum modulus and coefficients.

There is one interesting case when the function $B(r)$ completely determines the limit zero distribution μ .

Proposition. *Suppose that the function $z \mapsto B(|z|, u)$ is piecewise-harmonic in $\{z : |z| < R\}$ in the following sense: there exists a closed nowhere dense set $E \subset (0, R)$ such that for every component I of $(0, R) \setminus E$, we have*

$$B(r, u) = a_I \log r + b_I, \quad r \in I,$$

and $B(r, u) = \text{const}$ for sufficiently small r . Then $u(z) = B(|z|, u)$, $|z| < R$.

For example, if $B(r, u) = \log^+ r$ then $u(z) = \log^+ |z|$. We conclude that μ is the uniform distribution on the unit circle if and only if $B(r, u) = \log^+ |z|$. The corresponding functions are $\Phi(t) = t^+$, $-\infty < t < +\infty$, and $\phi(x) = 0$, $0 \leq x \leq 1$.

We obtain necessary and sufficient conditions, in terms of the coefficients, for the zeros to be uniformly distributed on the unit circle. For simplicity, we state it only for sequences of polynomials.

Corollary. *Let*

$$f_n = \sum_{k=0}^n a_{n,k} z^k$$

be a sequence of polynomials, and μ_n the empirical measures of f_n . Then μ_n converge weakly to the uniform probability measure on the unit circle, as $n \rightarrow \infty$, if and only if the following condition holds. For every $\epsilon > 0$ there exists n_ϵ , such that for every $n > n_\epsilon$ we have

$$\max_{k \in [0, n]} \log |a_{n,k}| - \max_{k \in [0, \epsilon n] \cup [(1-\epsilon)n, n]} \log |a_{n,k}| \leq \epsilon n. \quad (12)$$

An equivalent result was recently obtained by Fernandez [7].

For example, condition (12) holds if all coefficients are uniformly bounded from above, while $|a_{n,0}|$ and $|a_{n,n}|$ are bounded from below.

To prove the Proposition we notice that on any complementary interval I , $U(z) := a_I \log |z| + b_I$ is a harmonic majorant of u , and $u(z) = U(z)$ for some point inside the ring $\{z : |z| \in I\}$. So by the maximum principle $u = U$.

2 Applications of the previous results

Example 1. (Jentzsch) Let f be a power series with radius of convergence 1 and f_{n_k} a sequence of partial sums such that $|a_{n_k}|^{1/n_k} \rightarrow 1$. Then f_{n_k} satisfy

the assumptions of the Proposition, and we obtain a theorem of Jentzsch that zeros of f_{n_k} are asymptotically uniformly distributed on the unit circle.

Example 2. Asymptotic distribution of zeros of a Ruelle zeta-function [6]. This zeta function is related to the dynamical system $z \mapsto p_c(z) = z^2 + c$, where c is a real parameter. The zeta function is

$$F_c(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{p_c(0) \cdots p_c^{*n}(0)},$$

where p_c^{*n} is the n -th iterate of p_c . For $c < -2$, we have a real one-parametric family of entire functions. Zeros of these functions are eigenvalues of a Ruelle operator. Let $V(c)$ be the smallest positive integer n , for which $p_c^{*(n+1)}/p_c^{*n} \geq 36$. By analyzing Taylor's coefficients of F_c , it was shown in [6] that

$$\frac{1}{V(c)} \log M(F_c, r) \rightarrow \log^+(r/2), \quad c \rightarrow -2, \quad r > 0.$$

Therefore, the Proposition with $R = \infty$ implies that the limit distribution of zeros is the uniform measure on the circle $|z| = 2$.

Similar arguments apply to the entire function

$$H_a(z) = \frac{1}{a} \sum_{n=0}^{\infty} a^{2^n} z^n, \quad 0 < a < 1,$$

studied by Hardy [8]. As $a \rightarrow 1-$, the limit distribution of zeros is the uniform measure on the unit circle.

Example 3. The results about this example were first obtained by Alan Sokal and Alexander Scott (2005, unpublished).

Consider the polynomials

$$\bar{C}_n(y) = \sum_{k=n-1}^{n(n-1)/2} c_{n,k} (y-1)^k,$$

where $c_{n,k}$ is the number of simple connected graphs with k edges on n labeled vertices. Tutte [15] discovered the following identity for the the exponential generating function of \bar{C}_n :

$$\sum_{n=1}^{\infty} \bar{C}_n(y) x^n / n! = \log F(x, y), \tag{13}$$

where

$$F(x, y) = \sum_{n=0}^{\infty} y^{n(n-1)/2} x^n / n!.$$

This series F represents an entire function of x for $|y| \leq 1$. It is of order zero when $|y| < 1$, and thus it has roots. If $x_0(y)$ is the root of the smallest modulus, then the series in the left hand side of (13) is convergent for $|x| < |x_0(y)|$.

From the formula for the radius of convergence, we obtain

$$\limsup_{n \rightarrow \infty} n^{-1} \log |\overline{C}_n(y)/n!| = -\log |x_0(y)|, \quad |y| < 1.$$

It is not difficult to prove that actually the limit exists

$$\lim_{n \rightarrow \infty} n^{-1} \log |\overline{C}_n(y)/n!| = -\log |x_0(y)| \quad \text{for } |y| < 1. \quad (14)$$

This relation justifies the need to study the function $y \mapsto x_0(y)$. For a survey of known results and many conjectures about this function we refer to [14].

From the definition of \overline{C}_n we infer that these are monic polynomials of degree $d_n = n(n-1)/2$. So

$$\begin{aligned} u_n(y) &:= d_n^{-1} \log |\overline{C}_n(y)/n!| = d_n^{-1} \sum_{k=0}^{n(n-1)/2} \log |y - y_k| + c_n \\ &= \int \log |y - w| d\mu_n(w) + c_n, \end{aligned}$$

where μ_n is the empirical measure of \overline{C}_n , and $c_n \sim (\log n!)/d_n$ are constants tending to zero. From the combinatorial definition one can compute the values $\overline{C}_n(0) = (-1)^{n-1}(n-1)!$. Thus

$$\lim_{n \rightarrow \infty} u_n(0) = 0. \quad (15)$$

Equation (14) implies convergence $u_n \rightarrow u$, where u is the potential of the limit measure μ of empirical measures.

As the plane is not compact, the limit measure μ can a priori have smaller total mass than 1, but we will see that this does not happen in our situation: the zeros of \overline{C}_n do not escape to ∞ .

We have $u_n \rightarrow u$, where

$$u(y) = \int \log |y - w| d\mu(w), \quad \mu(\mathbf{C}) \leq 1. \quad (16)$$

Equation (14) implies that

$$u(y) \leq 0, \quad \text{for } |y| < 1. \quad (17)$$

On the other hand, (15) implies that $u(0) \geq 0$, so, by the Maximum Principle,

$$u(y) = 0 \quad \text{for } |y| < 1. \quad (18)$$

By a well-known property of subharmonic functions, this implies

$$u(y) = 0 \quad \text{for } |y| \leq 1. \quad (19)$$

On the other hand, from (16) follows that $u(y) \leq \log |y| + o(1)$ as $y \rightarrow \infty$, and together with (19) this implies that for every $\epsilon > 0$, the function $(1 + \epsilon) \log |y|$ is a harmonic majorant of u in the annulus $1 < |y| < \infty$. So we have

$$u(y) \leq \log^+ |y|, \quad \text{for all } y \in \mathbf{C}. \quad (20)$$

We already know that equality holds in the unit disc. To show that it holds everywhere, we produce a lower estimate as follows. From Cauchy's inequality

$$2^{d_n} \leq M(2, \overline{C}_n) := \max_{|y|=2} |\overline{C}_n(y)|,$$

and it follows that $M(2, u) \geq \log 2$. Thus we have equality in (20) at some point y_1 , with $|y_1| = 2$. By the Maximum Principle, applied to the ring $1 < |y| < \infty$ we have equality in (20) everywhere, that is

$$u(y) = \log^+ |y|. \quad (21)$$

This implies that $\mu = (2\pi)^{-1} \Delta u$ is the uniform distribution on the unit circle, in particular μ is a probability measure in \mathbf{C} as advertised above. Moreover, μ_n tends to the uniform distribution on the unit circle.

3 Asymptotic behavior of derivatives

In this section, D is an arbitrary region in the complex plane, and f_n analytic functions in D . Suppose that the sequence of subharmonic functions $(\log |f_n|)/n$ converges:

$$\frac{1}{n} \log |f_n| \rightarrow u.$$

Then one can choose a subsequence such that

$$\frac{1}{n} \log |f'_n| \rightarrow v.$$

We fix a sequence for which both limits hold.

Theorem 2. *Under the stated conditions, $v \leq u$, and if at some point $v(z_0) < u(z_0)$, then there is a neighborhood V of z_0 such that u is constant on V .*

Proof. For every subharmonic function w and $\epsilon > 0$ we define

$$w_\epsilon(z) = \max_{\zeta: |\zeta-z| \leq \epsilon} w(\zeta).$$

Then w_ϵ is a decreasing sequence of subharmonic functions, and for every z , $w(z) \leq w_\epsilon(z) \rightarrow w(z)$ as $\epsilon \rightarrow 0$.

Choose an arbitrary $z_0 \in D$. Cauchy's estimate gives

$$|f'_n(z)| \leq \epsilon^{-1} |f_n|_{2\epsilon}(z_0), \quad |z - z_0| < \epsilon.$$

Taking logs, dividing by n and passing to the limit, we conclude that

$$v_\epsilon(z_0) \leq u_{2\epsilon}(z_0).$$

This holds for every sufficiently small $\epsilon > 0$, so passing to the limit when $\epsilon \rightarrow 0$, we obtain $v(z_0) \leq u(z_0)$.

To prove the second statement of the Theorem, we fix $z_0 \in D$, and suppose that

$$u(z_0) - v(z_0) = 5\delta > 0. \tag{22}$$

By upper semi-continuity of v_0 , there exists a disc V_{2r} of radius $2r$ centered at z_0 , such that

$$v(z) < v(z_0) + \delta, \quad z \in V_{2r}.$$

this implies that there exists n_0 such that for $n > n_0$ we have

$$\log |f'_n(z)| \leq n(v(z_0) + 2\delta), \quad z \in V_r. \tag{23}$$

Consider the set

$$E = \{z \in V_r : u(z) \geq u(z_0) - \delta\}.$$

This set is non-empty because it contains z_0 , and it has positive area.

According to a theorem of Azarin [1], the set

$$\{z \in V_r : |u(z) - (\log |f_n|)/n| > \delta\}$$

can be covered by a countable set of discs with the sum of radii tending to 0 as $n \rightarrow \infty$. This means that for n sufficiently large, there exist points $z_n \in E$ such that

$$|f_n(z_n)| \geq n(u(z_0) - 2\delta). \quad (24)$$

Now for $z \in V_r$ we have in view of (23):

$$||f_n(z)| - |f_n(z_n)|| \leq |f_n(z) - f_n(z_n)| \leq 2re^{n(v(z_0)+2\delta)}.$$

This gives in view of (24) and (22) that

$$\log |f_n(z)| = \log |f_n(z_n)| + \log(1 + 2re^{-n\delta}) = \log |f_n(z_n)| + o(1),$$

which implies that u is constant in V_r .

Example 4. Let P_n be polynomials of degree n and μ_n their empirical measures. Suppose that $\mu_n \rightarrow \mu$ weakly, where μ is a Borel measure in the plane.

If the complement of the closed support of μ does not have bounded components, then the empirical measures μ' describing the distribution of critical points of P_n converge to the same measure μ .

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