AN EXTREMAL PROBLEM FOR POLYNOMIALS

A. EREMENKO AND L. LEMPERT

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Dedicated to Paul Erdős on his 80th Anniversary

ABSTRACT. Let $f(z) = z^n + \cdots$ be a polynomial such that the level set $E = \{z : |f(z)| \leq 1\}$ is connected. Then $\max\{|f'(z)| : z \in E\} \leq 2^{(1/n)-1}n^2$, and this estimate is the best possible.

The following question was asked in [1, Problem 4.8]. Let f be a polynomial of degree n,

$$(1) \qquad\qquad f(z) \sim z^n, \qquad z \to \infty.$$

Put $E_f = \{z : |f(z)| \leq 1\}$, and assume that E_f is connected. Is it true that

$$(2) \qquad\qquad \max\{|f'(z)| : z \in E_f\} \leq \tfrac{1}{2}n^2?$$

Pommerenke [2] proved this with $en^2/2$ instead of $n^2/2$. In a recent survey of unsolved problems [3] Erdős noticed that the estimate (2) has to be replaced by $\frac{1}{2}\{1 + o(1)\}n^2$ and suggested that Chebyshev polynomials should be extremal.

We will give an affirmative answer to this question. Let us first describe in detail the extremal polynomials for the problem. Define the Chebyshev polynomials T_n, $n = 1, 2, \ldots$, by the usual relation $\cos nz = T_n(\cos z)$, and set $f_n(z) = T_n(2^{(1-n)/n}z + 1)$. Then the polynomials f_n have the following properties:

(i) $f_n(0) = 1$.
(ii) $f_n'(0) = 2^{(1/n)-1}n^2$.
(iii) f_n is real and all its zeros are negative.
(iv) The critical values of f_n are equal to ± 1.

The polynomials f_n are uniquely characterized by properties (1), (i), (iii), and (iv). To prove this assume that a polynomial f of degree n satisfies (1), (i), (iii), and (iv). Then all zeros of f and f' are real and simple. There are exactly $n - 1$ points $z < 0$ such that $f(z) = \pm 1$ and $f'(z) = 0$. Choose real constants a and b such that the polynomial $g(z) = f(az + b)$ has all its zeros (hence critical points) on $(-1, 1)$, $g(1) = 1$, and $g(-1) = (-1)^n$. Then the

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only poles of the rational function $R = g'^2/(1 - g^2)$ are 1 and -1, and we have

$$R(z) \sim -n^2 z^{-2}, \qquad z \to \infty.$$

We conclude that $g'^2/(1 - g^2) = n^2/(1 - z^2)$. The solution of this differential equation satisfying $g(1) = 1$ is the Chebyshev polynomial $T_n(z) = \cos(n \arccos z)$. Now using (1), (i), and (iii) we get $a = 2^{(n-1)/n}$ and $b = -2^{(n-1)/n}$, so $f = f_n$.

Observe that the conditions and the conclusion of the conjecture are invariant with respect to the substitution $f(z) \mapsto f(z + c)$, $c \in \mathbf{C}$, so the conjecture follows from

Theorem 1. *Let f be a polynomial satisfying* (1), $|f(0)| = 1$, *and E_f is connected. Then $|f'(0)| \leq f_n(0) = 2^{(1/n)-1}n^2$. Equality can occur only for $f(z) = c^{-n} f_n(cz)$, $|c| = 1$.*

Proof. It is evident that an extremal polynomial for the problem does exist. We have to prove only that it is equal to $c^{-n} f_n(cz)$, $|c| = 1$. Let f be an extremal polynomial. From $|f(0)| = 1$ follows

$$f(z) = \lambda \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right), \qquad |\lambda| = 1,$$

and (1) implies that $\prod |z_k| = 1$. Denote

$$f^*(z) = \prod_{k=1}^{n} \left(1 + \frac{z}{|z_k|}\right).$$

Then f^* satisfies (1) and $f^*(0) = 1$. The condition that E_f is connected implies

$$m(r, f) := \min\{|f(z)| : |z| = r\} \leq 1, \qquad 0 \leq r \leq \max_k |z_k|.$$

It is clear that $|f^*(-r)| \leq m(r, f)$, $r \geq 0$, because $|1 - r/|z_k|| \leq |1 - re^{i\theta}/z_k|$ for all k and θ, so

(3) $|f^*(-r)| \leq 1, \qquad 0 \leq r \leq \max_k |z_k|.$

Thus the zeros of f^* can be connected within E_{f^*}. The minimum principle therefore implies that E_{f^*} is connected. Thus f^* satisfies the conditions of the theorem. We have

$$(f^*)'(0) = \sum_{k=1}^{n} \frac{1}{|z_k|} \geq \left|\sum_{k=1}^{n} \frac{1}{z_k}\right| = |f'(0)|,$$

and equality holds if and only if all z_k have the same argument. So f^* is also extremal, and $f(z) = c^{-n} f^*(cz)$, $|c| = 1$.

It remains to prove that $f^* = f_n$. By Rolle's theorem all critical points of f^* lie on the segment $[-A, -a]$, where $A = \max|z_k|$ and $a = \min|z_k|$. By (3) all critical values of f^* are between 1 and -1. Denote by $0 > a_1 > a_2 > \cdots > a_m$ all critical points where critical values are equal to ± 1. If $m = n - 1$ then $f^* = f_n$ and we are done.

Now assume that $m \leq n - 2$. Put $a_0 = 0$, and construct a real polynomial p of degree at most m such that

(4) $$p(a_k)f^*(a_k) > 0, \qquad 0 \leq k \leq m.$$

To do this it is enough to place one zero of p between a_k and a_{k+1} for each k such that $f^*(a_k)$ and $f^*(a_{k+1})$ have opposite signs. The sequence $a_0, \cdots, a_m$ has at most m changes of sign, so the degree of p is at most m. For a small number $\varepsilon > 0$ consider the polynomial

$$P(z) = f^*(z) + \varepsilon z p(z),$$

which satisfies (1) and $P(0) = 1$. We prove that the set E_P is connected if ε is small enough. First we have $|P(z)| \leq 1$, $-A \leq z \leq 0$, for small ε in view of (4) with $1 \leq k \leq m$. Second, denote by N a complex neighborhood of zeros of f^*, such that $|f^*(z)| < \frac{1}{2}$, $z \in N$. If ε is small enough then all zeros of P belong to N and also $|P(z)| < 1$, $z \in N$. So we have $|P(z)| \leq 1$ when z belongs to the connected set $N \cup [-A, -a]$, containing all zeros of P. This implies connectedness of E_P.

So P satisfies all conditions of the theorem. On the other hand, (4) with $k = 0$ implies that $P'(0) > (f^*)'(0)$, which contradicts the extremality of f^*. This proves the theorem.

REFERENCES

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907
E-mail address: eremenko@math.purdue.edu
E-mail address: lempert@math.purdue.edu