$$\forall\beta\in W^1_{b,0}(A_n'):\beta x\in W^1(H)\subset\mathcal{D}_0.$$

By choosing a suitable sequence of random elements $\{\beta_n x;\ n\geq 1\}$, we see that Lemma 3 holds.

Proof of the Assertions in Example 2. Using the random variable $\alpha$ we can construct a sequence of random variables $\{\beta_n=\varphi_n(\alpha),\ \varphi_n\in C^{1,b};\ n\geq 1\}$ such that

1) $\beta_n=\frac{1}{\alpha}$ for $\alpha>\frac{1}{n}$; $n\geqslant 1$;

2) $n+1\geqslant\beta_n>0\ (\mathrm{mod}\,P)$, $n\geqslant 1$;

3) $\beta_n\in W^1$, $\|D\beta_n\|\in L_\infty(\Omega,\mathcal{F},P)$.

By considering the sequence $\{\beta_n\alpha x;\ n\geq 1\}$ we have $x\in\mathcal{D}$. In addition, for each $n\geq 1$, integrating by parts we obtain

$$(\beta_n\alpha x;\xi)=\beta_n\langle\alpha x;\xi\rangle+(\alpha x;D\beta_n).$$

On the set $\left\{\alpha>\frac{1}{n}\right\}\in\mathcal{E}:D\beta_n=-\frac{1}{\alpha^2}D\alpha$. Consequently,

$$(x;\xi)=\lim_{n\to\infty}(\beta_n\alpha x;\xi)=\frac{1}{\alpha}\langle\alpha x;\xi\rangle-\frac{1}{\alpha}(x;D\alpha)\quad(\mathrm{mod}\,P).$$

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# PERIODIC POINTS OF POLYNOMIALS


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We recall the most important facts of the Julia–Fatou theory as related to the iteration of polynomials (cf. [1-3]). Let P be a polynomial of degree $m\geq 2$, and $P^n$ its n-th iteration. A point z is called periodic if $P^n z = z$ for some $n\in\mathbb{N}$. The set $\{p^k z\}_{k=1}^n$ is then called a cycle, and its cardinality is called the order of the cycle. The number $\lambda=(P^n)'(z)$, where $z\neq\infty$, is called a multiplicator of a cycle of order n. A cycle is called repulsive if $|\lambda|>1$. Let $D_\infty=\{z\in\overline{\mathbb{C}}:P^n z\to\infty,\ n\to\infty\}$. It is easy to see that $D_\infty$ is a region and that $\infty\in D_\infty$. The boundary of this region is called the Julia set $J=J(P)$. An equivalent definition is as follows. Let N(P) be the largest open set in $\mathbb{C}$ on which the family $\{P^n\}$ is normal. Then $J(P)=\mathbb{C}\setminus N(P)$. The Julia set is perfect and fully invariant, i.e., $P^{-1}(J)=J$. Furthermore, $J(P^n)=J$, $n\in\mathbb{N}$. The polynomial P has no more than $m-1$ nonrepulsive cycles [2]. On the other hand, the number of repulsive cycles is infinite; their union is a dense subset of J.

Let D be a region, and $z_0\in\partial D$. A point $z_0$ is called attainable (from D) if there exists a curve $\Gamma\subset D$ which ends on $z_0$.

THEOREM 1. Repulsive periodic points of the polynomial P are attainable from $D_\infty$.

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In the case where the set J(P) is connected, Theorem 1 was announced by Douady [2]. As far as we know, the proof was not published.

Denote by $T_m$ a polynomial defined by the functional equation $\cos m\omega = T_m(\cos\omega)$. The Julia set $J(T_m)$ is the set $[-1, 1]$. If $R_m(z) = z^m$, then $J(R^m)$ is the unit circle. The polynomials $T_m$ and $R_m$ play an extremely important role in iteration theory [1, 3].

THEOREM 2. Suppose the set J(P) is connected. Then the equation

$$|\lambda| \leqslant m^{2n}, \quad m = \deg P \tag{1}$$

for the multiplicator $\lambda$ holds for any cycle of order n. Equality is achieved in Eq. (1) if and only if P is conjugate to $T_m$ by a linear transformation, and $\lambda$ is the multiplicator of the endpoint of the interval J(P) (which is a fixed point).

Note that if Theorem 1 is proved, then Eq. (1) (without the case of equality) follows from Theorem 3 of Pommerenke's paper [4].

THEOREM 3. For any polynomial P of degree m, one of the following is true:

1) there exists a cycle of order n with multiplicator $\lambda$, such that $|\lambda| > m^n$;

2) P is conjugate to $R_m$ by a linear transformation.

Note that it is enough to prove Theorems 1 and 2 for fixed points, i.e., cycles of order 1.

The proof of Theorems 1 and 2 is cased on the study of the entire function introduced by Poincaré. Suppose $P(z_0) = z_0$, $P'(z_0) = \lambda$, $|\lambda| > 1$. From Poincaré's theorem [1], the functional equation

$$f(\lambda z) = P(f(z)) \tag{2}$$

has an entire solution f; moreover, this solution is uniquely determined by the conditions

$$f(0) = z_0, \quad f'(0) = 1. \tag{3}$$

[The simplest proof of these facts (it belongs to Poincaré) goes as follows: we first determine the formal power series $f(z) = z_0 + z + c_2z^2 + \ldots$ satisfying Eq. (2), and then show, by a direct analysis of the coefficients, that the series converges in some neighborhood of zero; finally, we extend the function f into $\mathbb{C}$ with the help of Eq. (2), taking into account that $|\lambda| > 1$.]

Denote by I the set of points on which the family $\{f(\lambda^n z) : n \in \mathbb{N}\}$ is not normal. It is clear that $I = f^{-1}(J)$. Set $D = f^{-1}(D_\infty)$. It follows from Eq. (2) and the full invariance of J and $D_\infty$ that

$$\lambda I = I, \quad \lambda D = D. \tag{4}$$

Let G be the Green's function for the region $D_\infty$ with a pole in $\infty$ which is continued by the zero function on $\mathbb{C}\setminus D_\infty$. The function G is continuous and subharmonic in $\mathbb{C}$ and obeys the functional equation found in [2]:

$$G(P(z)) = mG(z), \quad z \in \mathbb{C}. \tag{5}$$

[This property follows immediately from the evident relation $G(z) = \lim\limits_{n\to\infty} m^{-n} \ln |P^n(z)|$.]

The function $u(z) = G(f(z))$ is continuous and subharmonic in $\mathbb{C}$. It follows from Eqs. (2) and (5) that

$$u(\lambda z) = mu(z), \quad z \in \mathbb{C}. \tag{6}$$

Subharmonic functions satisfying Eq. (6) play an important role in the theory of entire functions (see, for example, [5]).

The order of a subharmonic function u in $\mathbb{C}$ is determined by the formula $\rho = \overline{\lim\limits_{r\to\infty}} \ln B(r, u)/\ln r$, where $B(r, u) = \max\{u(z) : |z| = r\}$. It follows from (6) that

$$\rho = \ln m / \ln |\lambda|. \tag{7}$$

According to the "subharmonic version of the Denjoy—Carleman—Ahlfors theorem" [6, Theorem 4.16], the number of connected components of the set $\{z : u(z) > 0\}$ does not exceed $\max\{2\rho, 1\}$. We denote these components by $D_1, \ldots, D_p$. From Eq. (4) there exists an N such that $\lambda^N D_1 = D_1$. Choose a point $\omega_0 \in D_1$ and connect it by a curve $\lambda_0 \subset D_1$ with the point $\lambda^{-N}\omega_0 \in D_1$. Then $\Gamma = \bigcup_{n=0}^{\infty} \lambda^{-Nn}\Gamma_0 \subset D_1$ is a curve which goes to zero. Its image $f(\Gamma) \subset D_\infty$ is a curve which goes to $z_0$ by virtue of Eq. (3). This proves Theorem 1.

To prove Theorem 2, we note that if the Julia set is connected, then the set $I = f^{-1}(J)$ contains the continuum K connecting 0 and $\infty$. Indeed, if this is not true, then there exists a closed Jordan curve $\gamma$ which separates 0 and $\infty$; moreover, $\gamma \cap I = \phi$. Let V be a neighborhood of zero on which f is bijective [this exists by virtue of Eq. (3)]. Choose V small enough so that J is not contained in f(V). Let M be a number large enough so that $\lambda^{-M}\gamma \subset V$. Taking Eq. (4) into account, we find $\lambda_{-M}\gamma \cap I = \phi$. Thus $f(\lambda^{-M}\gamma) \cap J = \phi$ and the curve $f(\lambda^{-M}\gamma)$ separates $z_0$ and $\infty$. This contradicts the fact that the set J is connected.

A classical theorem of Wiman (see, for example, [5]) states that for a harmonic function v of order $\rho < 1$, the inequality

$$\overline{\lim_{r\to\infty}} A(r, v)/B(r, v) \geqslant \cos \pi\rho,$$

holds, where $A(r, v) = \inf\{v(z) : |z| = r\}$.

Since $u(z) = 0$, $z \in K$, we have that $A(r, u) \equiv 0$, and hence $\rho \geq 0.5$. The inequality (1) (with $n = 1$) now follows from Eq. (7).

Suppose now that equality holds in Eq. (1). Then $\rho = 0.5$. We show that the subharmonic function $u \geq 0$ of order 0.5 satisfying conditions (6) and $A(r, v) \equiv 0$ necessarily takes the form

$$u(re^{i\theta}) = cr^{1/2}\cos\frac{1}{2}(\theta-\theta_0), \quad |\theta| \leqslant \pi, \tag{8}$$

where $c > 0$ and $\theta_0 \in [-\pi, \pi]$ are some constants. This result may be derived from [7, 8]; nonetheless, we present an independent simple proof.

The function u can be represented as [9]

$$u(z) = \int_{\mathbb{C}} \ln\left|1 - \frac{z}{\zeta}\right| d\mu_\zeta,$$

where $\mu$ is some Borel measure. Let $n(t) = \mu\{\zeta : |\zeta| \leq t\}$. Then $\rho = \lim_{t\to\infty} \ln n(t)/\ln t$. Let

$$u^*(z) = \int_0^\infty \ln\left|1 - \frac{z}{t}\right| dn(t).$$

The subharmonic function $u^*$ is of order $\rho$. We show that the measure $\mu$ is concentrated on the ray $\ell = \{\zeta : \arg\zeta = \theta_0\}$. Suppose this is not so. Taking into account the fact that for fixed $r > 0$ the quantity $\ln|1 - re^{i\theta}|$ has a strict minimum for $\theta = 0$, we obtain

$$u^*(r) = A(r, u^*) < A(r, u) = 0, \quad r > 0.$$

Then it follows from Eq. (6) that $u^*(|\lambda|z) = mu^*(z)$, and hence

$$\overline{\lim_{r\to\infty}} A(r, u^*)/B(r, u^*) < 0.$$

This contradicts Wiman's theorem.

Thus the measure $\mu$ is concentrated on some ray $\ell$, and the function u is harmonic in $\mathbb{C}\setminus\ell$. Since $u \geq 0$, the inequality $u > 0$ holds in $\mathbb{C}\setminus\ell$. Moreover, $u = 0$ on $\ell$, since $A(r, u) \equiv 0$. Thus, u has the form of Eq. (8).

From this it follows that I is a ray. Since $I = f^{-1}(J)$, there exists a circle V such that $J \cap V$ is an analytic curve. Then it follows from Fatou's theorem [1, p. 225] that the polynomial P is conjugate to either $T_m$ or $R_m$. The latter case is eliminated by direct checking. Theorem 2 is proved.

Remark. Let G be a region, and let $z_0 \in \partial G$ be an attainable boundary point. Two curves $\Gamma_1, \Gamma_2 \subset G$ ending on the point $z_0$ are called equivalent if there exists a sequence of curves $\gamma_n \subset G$, $\gamma_n \to z_0$ connecting $\Gamma_1$ and $\Gamma_2$. From the results of Douady [2, Sec. 6, Lemma 1], it follows that if the Julia set is connected, then there exists a finite number of classes of equivalent curves in $D_\infty$ which end on the periodic point $z_0$. It is possible to show that the number p of these classes is equal to the number of connected components of the set $D = \{z : u(z) > 0\}$. Applying the Denjoy–Carleman–Ahlfors theorem, we find $p \le 2\rho$. From Eq. (7) we find that $|\lambda| \le m^{2/p}$ for a fixed point, or $|\lambda| \le m^{2n/p}$ for a cycle of order n. More delicate arguments show that the equality in these estimates is possible in only two cases: 1) p = 1; P is conjugate to $T_m$ and $z_0$ is the endpoint of the interval J(P); 2) p = 2; P is conjugate to $T_m$ and $z_0$ is an interior point of the interval J(P).

We now go to the proof of Theorem 3. Without loss of generality, we can assume that the leading coefficient of the polynomial P is equal to 1. This is always possible to achieve by conjugating with a linear function which does not change the multiplicators. We will need the following lemmas.

LEMMA 1. Let

$$c = c(A) = \sum_{P(z)=A} P'(z).$$

Then c does not depend on A. Moreover,

$$\sum_{P^n(z)=A} (P^n)'(z) = c^n, \quad n \in \mathbb{N}, \tag{9}$$

$$\sum_{P^n(z)=z} (P^n)'(z) = m^n(m^n - 1) + c^n, \quad n \in \mathbb{N}, \tag{10}$$

$$m = \deg P.$$

Proof. From the residue theorem

$$c(A) - c(B) = \int_{|z|=r} \left\{ \frac{(P')^2}{P-A} - \frac{(P')^2}{P-B} \right\} dz,$$

where r is sufficiently large. The expression in the integral is $O(z^{-2})$, $z \to \infty$; hence c(A) = c(B). We prove Eq. (9) by induction:

$$\sum_{P^{k+1}(z)=A} (P^{k+1})'(z) = \sum_{P^k(\omega)=A} (P^k)'(\omega) \sum_{P(z)=\omega} P'(z) = c \sum_{P^k(\omega)=A} (P^k)'(\omega).$$

In view of Eq. (9), it is enough to prove Eq. (10) for n = 1. Then Eq. (10) follows from the fact that the residue of the function

$$\frac{P'(z)(P(z)-z)'}{P(z)-z} - \frac{(P')^2(z)}{P(z)}$$

in the point $\infty$ is equal to $m(m-1)$. The lemma is proved.

LEMMA 2. Suppose all the fixed points of the polynomial P, with the possible exception of one, have multiplicators equal to m = deg P. Then P is conjugate to $R_m$.

Proof. By conjugating with the function $z + a$, we make the exceptional point be equal to 0. From the hypotheses of the lemma, it follows that $P(z) - z = z/m(P'(z) - m)$. Solving this differential equation with the boundary condition P(0) = 0, we find that $P = R_m$.

LEMMA 3. Let $c, \lambda \in \mathbb{C}$. If $\mathrm{Re}(\lambda^n) > \mathrm{Re}(c^n)$ for all $n \in \mathbb{N}$, then $\lambda > 0$.

Proof. The cases $c\lambda = 0$, $\arg c = \pm\pi/2$, $\arg\lambda = \pm\pi/2$ are easily eliminated. There exists a $\delta > 0$ such that infinitely many points $c^n$ lie within the angle $\{z : |\arg z| \le \pi/2 - \delta\}$.

This implies that $|\lambda| \geq |c|$. If $\arg\lambda \neq 0$, then infinitely many points $\lambda^n$ lie within some angle of the form $\{z : |\arg z - \pi| \leq \pi/2 - \delta\}$. Thus $|\lambda| = |c|$. Setting $\theta_1 = \arg\lambda$, $\theta_2 = \arg c$, we obtain

$$\cos n\theta_1 > \cos n\theta_2, \quad n \in \mathbb{N}. \tag{11}$$

In particular, $\cos 2n\theta_1 > \cos 2n\theta_2$, which implies

$$\cos^2 n\theta_1 > \cos^2 n\theta_2, \quad n \in \mathbb{N}. \tag{12}$$

It follows from Eqs. (11) and (12) that $\cos n\theta_1 > 0$, $n \in \mathbb{N}$. Thus $\theta_1 = 0$, which is shat was needed.

We now finish the proof of Theorem 3. Suppose that the moduli of the multiplicators of all cycles do not exceed $m^n$, where n is the order of the cycle.

Let $\lambda$ be the multiplicator of any fixed point. Then, by assumption,

$$\operatorname{Re} \sum_{P^n(z)=z} (P^n)'(z) \leqslant m^n(m^n - 1) + \operatorname{Re}(\lambda^n); \tag{13}$$

moreover, the equality holds only if the multiplicators of all the fixed points, with the exception of one, are equal to m. Then from Lemma 2 we find that P is conjugate to $R_m$. Suppose that the inequality (13) is strict for all $n \in \mathbb{N}$. Comparing Eqs. (13) and (10), we find that $\operatorname{Re}(\lambda^n) > \operatorname{Re}(c^n)$, $n \in \mathbb{N}$. From Lemma 3, $\lambda > 0$. This holds for all fixed points. If all their multiplicators are equal to m, we apply Lemma 2 again. Suppose that the equation $\lambda_i < m - \varepsilon$, $i = 1, 2$, $\varepsilon > 0$ holds for two multiplicators. Choose a sequence $n_k$ such that $\operatorname{Re}(c^{n_k}) \geq 0$, $k \in \mathbb{N}$ is satisfied. By virtue of Eq. (10), we find that either $m^{n_k}(m^{n_k} - 1 \leqslant \sum_{P^{n_k}(z)=z} \operatorname{Re}(P^{n_k})'(z) \leqslant m^{n_k}(m^{n_k} - 2) + 2(m - \varepsilon)^{n_k}$ or $m^{n_k} \leq 2(m - \varepsilon)^{n_k}$, which is impossible. Theorem 3 is proved.

Remark. Suppose P(z) is a polynomial of degree $m \geq 2$ whose Julia set is connected. Denote by K(P) the lower bound of those $x > 0$, for which the inequality $|\lambda| \leq m^{nx}$ is satisfied for multiplicators $\lambda$ of all cycles of order $n \in \mathbb{N}$. We have proved that $1 \leq K(P) \leq 2$. By the method of extremal lengths it is possible to prove the strict inequality $K(P) < 2$ for the case in which the mapping $P : J(P) \to J(P)$ is hyperbolic [3].

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