A letter to V. Ya. Lin on a one-parametric family of differential equations

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Dear Volodya:

A year ago I told you about the following result [1]. For every differential equation $F(y', y, z) = 0$, where F is a polynomial, there exists an effective bound of degrees of rational solutions $y(z)$. This bound depends not only of degree of F but on the arithmetic of coefficients. The only example of a one-parametric family with rational solutions of arbitrarily high degree I had at that time was $zy' = \lambda y$ which has solutions z^λ.

Then you made a conjecture that the bound of degrees of solutions depends on the "size" of coefficients, rather than the arithmetic. That is if we solve the equation with respect to y' and get $y' = R(y, z, \lambda)$, where R is an algebraic function, than the values of parameter leading to rational solutions of high degree are such that $R(y_0, z_0, \lambda)$ becomes arbitrarily large for these values of lambda and some fixed pair (y_0, x_0).

I think if your conjecture were true, my result would have some simple proof. Now I can show with a simple example that your conjecture is not true (and thus my result is probably non-trivial).

Example.

$$\left(\frac{dy}{dz}\right)^2 = \frac{4y^3 + G_1 y + G_0}{4z^3 + g_1 z + g_0}.$$

It has a solution

$$y = \wp\left(\int_0^z \frac{dz}{\sqrt{4z^3 + g_1 z + g_0}}\right) := \wp(a(z)),$$

where $\wp$ is the Weierstrass function with certain lattice Γ of periods. Now the Abelian integral $a(z)$ also has a lattice of periods, which we denote by γ and the solution is rational if and only if $\gamma \subset \Gamma$. We have the following formulas

$$g_1 = 60 \sum \frac{1}{(n\omega_1 + m\omega_2)^4} \quad \text{and} \quad g_0 = 140 \sum \frac{1}{(n\omega_1 + m\omega_2)^6},$$

where the summation is over the whole lattice, except zero. Assume that $\gamma \subset \Gamma$ for certain choice of ω_j. If we leave ω_1 fixed and replace ω_2 by $k\omega_2$, where k is a positive integer, the new γ will still be a sublattice of Γ. The degree of our rational solution evidently tends to infinity when $k \to \infty$. (This degree by the way is the ratio of the areas of fundamental parallelograms of γ and Γ). On the other hand, when $k \to \infty$, both coefficients g_1 and g_0 tend to finite non-zero limits, namely

$$g_1^* = 60 \sum \frac{1}{n^4 \omega_1^4} \quad \text{and} \quad g_0^* = 140 \sum \frac{1}{n^6 \omega_1^6}.$$

Remarks. For the limit values of parameters the equation has a meromorphic transcendental solution in the whole plane.

Best regards, hope to see you soon (if I get my new pseudopassport in time to be able to come this winter).

References

[1] A. Eremenko, Rational solutions of first-order differential equations, Ann. Acad. Dci. Fenn. Math., V. 23, 1998, 181-190.

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