RESULTS FROM PRIOR NSF SUPPORT

A. Eremenko was supported by NSF grant DMS-1361836 (funding period 06/01/2014–05/31/2017). The project was titled "Problems in geometric function theory", and the amount of support was $ 172,904. The results of this research are contained in the 17 papers listed below (11 published, 2 accepted and 4 submitted).

A. Gabrielov, with Co-PI S. Basu, was supported by NSF grant DMS-1161629, (funding period 08/15/2012–07/31/2017). The project was titled "Semi-monotone sets and triangulation of definable families", and the amount of support was $ 300,000. The results of this research are contained in 17 papers, with Gabrielov being a co-author in 13 of them (8 published, 3 accepted and 2 submitted). Only those results of this project which are relevant to the current proposal are listed below.

Intellectual merit. The numbers in front of the paragraphs refer to the list of publications resulting from the NSF support at the end of this section.

1-5. In [21, 22, 23, 17, 18], the authors investigate the existence of conformal metrics of constant positive curvature on the sphere with $n \geq 4$ prescribed conical singularities. The similar problem with metrics of non-positive curvature was solved in the 19th century by E. Picard. Definitions and general background are given in Project Description. The difficulty of the problem depends on the number of conical singularities with non-integer angles. Such metrics satisfying the symmetry condition have been classified in the following cases: two non-integer angles [21, 22], three non-integer angles [23], and $n = 4$ with all angles half-integer [17, 18].

6. The problem of classification of metrics of positive curvature with one conical singularity on a torus was recently reduced by Chang-Shou Lin and his collaborators [34] to a question about Green's function. Green's function on a torus is defined as a solution of the equation $\Delta u = -\delta_0 + \text{const}$, where δ_0 is a delta-function, and the question is how many critical points can it have. The answer was obtained in [34]: 3 or 5, depending on the modulus of the torus. Their argument is extremely complicated and uses advanced non-linear PDE theory. We obtained a simpler proof by using elementary holomorphic dynamics. Relevance of holomorphic dynamics to this question is quite unexpected. Our proof gives an explicit description of the two regions on the moduli space of tori where the number of critical points is constant. An unexpected outcome of this argument is an example of a one-dimensional holomorphic dynamical system whose parameter space

consists of just two hyperbolic components and an analytic curve separating them.

7. The goal of the paper [11] was to improve the Second Main Theorem (SMT) of H. Cartan on holomorphic curves in complex projective space. Since the inception of Nevanlinna theory it has been speculated that the SMT must be an asymptotic equality, rather than an inequality. For the original SMT of Nevanlinna (that is in dimension 1), such an improvement was achieved by K. Yamanoi [46]. In this paper, the PI slightly generalizes Cartan's SMT and shows that indeed an asymptotic equality holds in a broad class of holomorphic curves, defined by a certain regularity condition. It is conjectured that one can get rid of the extra regularity condition, and that the modified SMT proposed in this paper holds as an asymptotic equality for all holomorphic curves.

8-9. The papers [2] and [24] contain an answer to the question of O. Zeitouni and S. Ghosh [26] on asymptotic distribution of zeros of polynomials with positive coefficients. The question arises in the study of the Law of large deviations for random polynomials. In the first paper we completely characterize the so called "empirical measures" of polynomials with positive coefficients. In the second paper we improve the classical inequality of Obreschkoff (1923) on the distribution of zeros of polynomials with positive coefficients and show that our result is best possible.

10. In [12], the PI gives a short proof of the so-called BMV conjecture previously established by H. Stahl [39]. This conjecture comes from mathematical physics, and states that if A, B are two Hermitian matrices and B is positive definite, then the function $t \mapsto \mathrm{trace}\, exp(A + Bt)$ is a Laplace transform of a positive measure.

11. A domain D with smooth boundary in the plane is called exceptional if there is a positive harmonic function in D which is zero on the boundary and whose normal derivative is constant on the boundary. Such domains were recently completely classified by Khavinson, Lundberg, Theodorescu and Traizet [44]. In [25], we relax the condition by requiring that the normal derivative is constant on each component of the boundary, and obtain a preliminary classification of such domains by reducing the problem to a problem about Abelian integrals. We also find new examples. Such domains are related to several problems on quadrature domains, minimal surfaces and fluid dynamics. E. Lundberg was a postdoc at Purdue University in 2011–2014.

12-13. In [4] we disprove the 34 years old conjecture of S. Bank and I. Laine on the zeros of solutions of the differential equation $w'' + Aw = 0$, where A is entire and transcendental. The conjecture was that the exponent of convergence of zeros of the product of two linearly independent solutions must be infinite unless the growth order of A is an integer. The conjecture has attracted a considerable interest; it was generally believed that it is true, but it has been proved only under very strong additional conditions on A. We employ a novel geometric technique to investigate the problem, and construct counterexamples. In the second paper on this subject, [5] we extend our technique, and show that all principal partial results on the conjecture are actually best possible.

14. In [6] we address the question asked by G. Gundersen [27]: does there exist an entire function with infinitely many zeros and infinitely many 1-points, such that zeros lie on the positive ray, and 1-points lie on two other rays? Among other things, we show that if such a function exists, the "other rays" must be of the form $\{z : \arg z = \pm\alpha\}$, where $0 < \alpha < \pi/2$, but the most surprising result is that such functions indeed exist when $\alpha \in (0, \pi/3]$ and when $\alpha = 2\pi/5$. It turns out that the problem is closely related to a class of functional equations satisfied by Stokes' multipliers of some linear differential equations. The same functional equations arise in the integrable models of statistical mechanics [9]. It is a challenge to find out what happens for the remaining range of α; this will be addressed in Project Description.

15. For general information on the Painlevé VI equation, see Project Description. A solution $w(z)$ is called exceptional if $w(z) \notin \{0, 1, z, \infty\}$ for $z \in \mathbf{C}\backslash\{0, 1\}$. In [19], we give a complete classification of exceptional solutions and the Painlevé VI equations that can have such solutions. This extends recent results in [7] where one special case is considered.

16–17. Paper [12] contains a discussion of some unsolved problems of real algebraic geometry and differential equations, based on the previous results of PI and co-PI. Paper [16] is a survey of the results of PI and co-PI from the research funded by NSF in 2011–2014.

Broader impact of the PI and co-PI activities. The results of PI found applications to the theory of random polynomials.

Human resources development. A. Eremenko served on PhD examination committees of

Alex Dyachenko (Technische Universität Berlin, January 2016),

Simon Albrecht (Christian-Albrecht Universität, Kiel, August 2015),

Samuel Roth (Indiana University–Purdue University in Indianapolis (IUPUI), 2015), and

Thomas Bothner (IUPUI, 2013).

A. Eremenko has taught two mini-courses for graduate students and young researchers: In Kent University in March 2015 (Informal Analysis

Seminar organized by F. Nazarov with a support from NSF, Lecture notes are posted on the web site of A. Eremenko at Purdue University. in Lviv University (Ukraine) in May 2016 (Lecture notes are in preparation). Both courses were based on the NSF-sponsored research of PI.

A. Eremenko continues collaboration with his former postdoc Eric Lundberg, and with his former PhD student Koushik Ramachandran.

A. Gabrielov advises a postdoctoral scholar Edinah Gnang, who was involved in some parts of the research funded by NSF. Dr. Gnang is completing his 3-year appointment at Purdue in Spring 2017, and has been offered a tenure-track assistant professor position at Johns Hopkins University starting July 2017. A. Gabrielov is currently serving on the PhD committees of K. Grady (Purdue), and D. Petrovic and B. Elwood (IUPUI).

Results of the recent joint work of A. Eremenko and A. Gabrielov were presented at several international conferences, including a lecture by A. Gabrielov at the Fields Institute in Toronto, Canada, on August 3, 2016, which was live-streamed, and a video has been posted on the Web site of the University of Toronto.

Enhancing infrastructure for research and education and increasing public scientific literacy and public engagement with science and technology.

Three of the publications listed above (6,11,17) are surveys, expositions or discussions of unsolved problems aimed at the general audience of mathematicians and physicists.

The PI maintains a web page, which contains various resources like lists of unsolved problems for beginning researchers, surveys, popular and advanced lectures, essays and problems on mathematical, scientific and engineering topics by the PI, list of free resources of mathematical literature, and so on. The PI receives a substantial feedback on this web page. Some materials from this web page were used in the scientific programs of the mass media.

The PI is active in the mathematical web sites MathOverflow and History of Science and Mathematics, which have very large and diverse set of participants. He writes Wikipedia articles on the subjects related to his research areas (user name: Pym1507).

Publications resulting from the NSF award.

1. A. Eremenko, A. Gabrielov and V. Tarasov, Metrics with conic singularities and spherical polygons, Illinois J. Math., 58, 3 (2014) 739-755.
2. A. Eremenko, A. Gabrielov and V. Tarasov, Metrics with four conic singularities and spherical quadrilaterals, Conformal Geometry and Dynamics, 20 (2016) 28–175.

3. A. Eremenko, A. Gabrielov and V. Tarasov, Spherical quadrilaterals with three non-integer angles, Journal of Math. Phys., Analysis and Geometry, 12 (2016) 2, 134-167.
4. A. Eremenko and A. Gabrielov, On metrics of curvature 1 with four singularities on tori and on the sphere, submitted, arXiv:1508.06510.
5. A. Eremenko and A. Gabrielov, Spherical rectangles, arXiv:1601.04060, accepted for publication in Arnold Math. Journal.
6. W. Bergweiler and A. Eremenko, Green functions and antiholomorphic dynamics on tori, Proc. AMS, 144, 7 (2016) 2911-2922.
7. A. Eremenko, On the second main theorem of Cartan, Ann. Acad. Sci. Fenn., 39 (2014) 895-871. Correction: vol. 40, 1 (2015).
8. W. Bergweiler and A. Eremenko, Distribution of zeros of polynomials with positive coefficients, Ann. Acad. Sci. Fenn., 40 (2015) 375-383.
9. A. Eremenko and A. Fryntov, Remarks on Obrechkoff's inequality, Proc. AMS, 144, 2 (2016) 703-707.
10. A. Eremenko, Herbert Stahl's proof of the BMV conjecture, Mat. Sbornik, 206, 1, (2015) 87–92.
11. A. Eremenko and E. Lundberg, Quasi-exceptional domains, Pacific J. Math., 276, 1 (2015) 167-183.
12. W. Bergweiler and A. Eremenko, On the Bank–Laine conjecture, accepted in J. European Math. Soc., arXiv:1408.2400.
13. W. Bergweiler and A. Eremenko, Quasiconformal surgery and linear differential equations, submitted, arXiv:1510.05731
14. W. Bergweiler, A. Eremenko and A. Hinkkanen, Entire functions with two radially distributed values, submitted, arXiv:1509.03283.
15. A. Eremenko, A. Gabrielov and A. Hinkkanen, Exceptional solutions to the Painlevé VI equation, submitted, arXiv:1602.04694.
16. A. Eremenko, Disconjugacy and secant conjecture, Arnold Math. J.,1 (2015) 3, 339-342.
17. A. Eremenko and A. Gabrielov, Spectral loci of Sturm–Liouville operators with polynomial potentials, In: Spectral Theory and Differential Equations. V. A. Marchenko 90th Anniversary Collection, Editors: E. Khruslov and L. Pastur, AMS Transl., 233 (2014) 135-143.

Evidence of research products and their availability. All 17 papers listed above are freely available on the arXiv. In addition, all of them are either published, or accepted for publication, or submitted for publication in the refereed mathematical journals.

PROJECT DESCRIPTION
Geometric methods in the analytic theory of differential equations

1. Real solutions of the Painlevé VI equation

Painlevé VI equation (PVI) is the following second order ODE:

$$\begin{aligned} q_{xx} &= \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-x}\right)q_x^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{q-x}\right)q_x \\ &+ \frac{q(q-1)(q-x)}{2x^2(x-1)^2}\left\{\kappa_4^2-\kappa_1^2\frac{x}{q^2}+\kappa_2^2\frac{x-1}{(q-1)^2}+(1-\kappa_3^2)\frac{x(x-1)}{(q-x)^2}\right\}, \end{aligned} \tag{1}$$

where $(\kappa_1,\ldots,\kappa_4)$ are parameters, and $q(x)$ is the solution. The first remarkable property of PVI is that all solutions are meromorphic in the twice punctured plane $\mathbf{C}\backslash\{0,1\}$, so it is an equation "without movable singularities". All equations of the form

$$y''=R(y',y,x) \text{ with } R \text{ rational in } y',y, \text{ meromorphic in } x,$$

with no movable singularities have been classified by Painlevé and Gambier in the beginning of 20th century; most of them can be reduced to linear ODE, and the rest to the six equations which are called Painlevé equations. Of those six, PVI is the most general one in the sense that the rest can be derived from it by certain limiting processes.

Independently of Painlevé and Gambier, and almost simultaneously, PVI was discovered by R. Fuchs, as the equation governing the isomonodromic deformation of a linear ODE

$$w''+P(z)w'+Q(z)w=0 \tag{2}$$

with five regular singularities at $0,1,x,q,\infty$, such that the exponent differences at the singularities $(0,1,x,\infty)$ are κ_j, while q is an apparent singularity with the exponent difference 2. If one varies x so that the monodromy of (2) remains unchanged, q must be a function of x, and this function must solve (1). All solutions of PVI are obtained in this way.

Since these discoveries of the early 20th century, the Painlevé equations have been intensively studied, and many important applications have been found in mathematics and physics. Their solutions, Painlevé transcendents, are gradually gaining the status of special functions.

Besides numerous applications, most work on PVI falls into three categories: a) algebraic transformations of the equation [38, 31], b) search for special solutions, like algebraic ones [35] or those expressed in terms of classical special functions [29], and c) asymptotic expansions at the fixed singularities $0, 1, \infty$, [32, 28].

We propose a new approach which is expected to give exact (non-asymptotic) information on a broad class of solutions. Specifically, we propose to study solutions of PVI with real parameters κ_j, which are real on one of the intervals between the fixed singularities $0, 1, \infty$. Without loss of generality, one may consider the interval $(1, \infty)$. Real solutions correspond to isomonodromic deformations of (2) where all parameters (singularities, exponent differences and accessory parameters) are real. The ratio $f = w_1/w_2$ of two linearly independent solutions of (2) satisfies the Schwarz differential equation

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = R(z), \tag{3}$$

where R is a real rational function with real poles. Equation (3) has 5 real singularities, and therefore f maps the upper half-plane H onto a circular pentagon spread over the Riemann sphere, which is defined up to a Möbius transformation. Notice that this pentagon is in general not a subset of the Riemann sphere, as its interior angles may be arbitrarily large, and some sides may be arbitrarily long. For the precise definition of a circular polygon see section 3. Such circular polygons have been studied in connection with linear differential equations by Klein [33] and his students [30], and recently in [37] and in our work [21, 22, 23, 17, 18].

The correspondence between real equations (3) with five real singularities and classes of circular pentagons modulo Möbius transformations is bijective. Thus every solution of PVI defines a one-parametric (parametrized by x) family of circular pentagons.

It turns out that the four interior angles of this pentagon are $\pi\kappa_j$, where κ_j are parameters of PVI, and the fifth angle is 2π. Moreover, our pentagon is of a special kind: it is a circular quadrilateral with a slit, and the tip of the slit is $f(q)$. (In some cases the quadrilateral degenerates into a triangle of digon.) Such special pentagons, modulo Möbius transformations are in bijective correspondence with real equations (3) with real singularities corresponding to PVI.

For a real solution $q(x)$ of PVI we call a point $x \in (1, +\infty)$ *special*, if $q(x) \in \{0, 1, x, \infty\}$. Special points are exactly those where the right hand side of (1) is not defined; by the result of Painlevé stated above, these points are poles or removable singularities of solutions.

We propose to solve the following problem:

For given real parameters κ_j and given real solution $q(x)$, $x \in (1, +\infty)$ to find how many special points it has and their mutual position.

So the outcome will be a sequence which shows in which order $q(x)$ takes the values $0, 1, \infty$ and x, as x runs over $(1, +\infty)$. We suppose that a simple combinatorial algorithm can be developed to solve this problem.

When x is a special point, our special pentagon degenerates to a circular quadrilateral without a slit (sometimes to a triangle or digon). We found transformation rules which relate the quadrilaterals corresponding to any two adjacent special points x_j, x_{j+1}. By applying these rules to any circular quadrilateral, we obtain a sequence of circular quadrilaterals which encodes the sequence of special points. This sequence can be either finite or infinite in one or both directions. This will give an algorithm of determining the sequence of special points. The problems that remain are:

a) How to construct a circular quadrilateral effectively from a given solution of PVI, and

b) What general properties of the sequence of special points can be derived from this algorithm.

Particular solutions of PVI are parametrized by conjugacy classes of the (projective) monodromy representations

$$(M_0, M_1, M_2, M_3) := (M_0, M_1, M_x, M_\infty), \quad M_j \in PSL(2, \mathbf{C})$$

of the associated equation (3). Real solutions correspond to a special class of monodromy representations with the property

$$M_j = \sigma_j \sigma_{j+1}, \quad j \in \mathbf{Z}_4, \tag{4}$$

for some reflections σ_j in the four circles to which the sides of our pentagons project. So the steps one has to perform to solve a) are:

To describe which monodromy representations satisfy (4), and how to find the reflections σ_j from the M_j effectively, and

To recover a circular pentagon from the parameters κ_j and the σ_j.

The first question seems to have a simple answer, while the second is difficult, especially when some inner angles $\pi\kappa_j$ are greater than 2π. We plan to begin with the case of unitary monodromy which is much better understood than the general case (see the next two sections).

Some new interesting general results can be obtained from the algorithm outlined above, for example

1. Classification of real solutions which have no real special points on $(1, +\infty)$.

2. A criterion which tells whether the sequence of special points is finite, infinite in one direction, or infinite in both directions.

Solutions of PVI with complex parameters without special points in $\mathbf{C}\backslash\{0, 1\}$ have been recently completely classified in our work [19].

An alternative aproach to problem 2 would use the Jimbo's asymptotics [32, 28] for solutions near 1 and $+\infty$ in terms of the monodromy representation. It will be interesting to compare the two approaches.

The problems described in this section are closely related to the problem of classification of circular quadrilaterals [30, 21, 22, 23, 17, 18] to which we return in the next two sections.

2. Metrics of curvature 1 with conical singularities

These Riemannian metrics are defined by the length element $ds = \rho(z)|dz|$, where z is a local conformal coordinate on a Riemann surface S. Here

$$\Delta \log \rho + \rho^2 = 2\pi \sum_{j=1}^{n} \alpha_j \delta_{a_j}. \tag{5}$$

The question is how many such metrics exist for given a_j and $\alpha_j > 0$. The analogous problem for non-positive constant curvature was solved by Picard in the end of 19th century: the only obstacle to solvability of (5) is the Gauss–Bonnet theorem, and if its conditions are satisfied, the metric exists and is unique (up to a constant multiple in the case of zero curvature) [40, 45]. The case of positive curvature is much more involved, and at present one has to restrict to special cases.

Equation (5) is a simplest representative of a class of non-linear PDEs which are called the mean field equations, and which are subject of current intensive research (see [43, 34, 7, 37]). Under the restriction $0 < \alpha_j < 1$, very complete results are available [45, 36].

We mostly restrict ourselves to the case when S is the sphere. In an important *symmetric case*, there is a circle $C \subset S$ which contains all singularities a_j, and the metric is symmetric with respect to C.

The problem has been completely solved in the case of three singularities [10], so the simplest unsolved case is four singularities. As every surface of constant curvature 1 is locally isometric to a piece of the standard sphere, we obtain the multi-valued *developing map*

$$f : S\backslash\{a_1, \ldots, a_n\} \to \overline{\mathbf{C}},$$

whose monodromy is a subgroup of $SU(2)$. Here $\overline{\mathbf{C}}$ is the sphere equipped with the standard spherical metric. This map f solves the Schwarz differential equation of the form (3), where R is a rational function with poles of second order at a_j, and the principal parts at these poles depending on α_j. The remaining parameters of R are essentially the accessory parameters which have to be determined so that the monodromy of f is unitarizable, that is conjugate to unitary monodromy. The Schwarz equation is equivalent to a linear ODE of the form (2), and our problem is equivalent to the following analytic problem:

For equation (2) with prescribed singularities and exponent differences, in how many ways one can determine the accessory parameters, so that the projective monodromy group is conjugate to a subgroup of $SU(2)$?

In the case of four singularities the linear equation (2) is the Heun equation, which has one accessory parameter λ. As one can take $(0, 1, a, \infty)$ as its singular points, our problem can be stated as counting solutions of one equation $F(a, \lambda) = 0$ for a given a. When this equation is algebraic, counting all solutions is usually not difficult, but to solve the symmetric problem we have to count its *real* solutions. However in most cases this equation is transcendental.

In our previous work [15, 20, 21, 22, 17, 18] we developed two approaches to this problem:

A geometric approach which works in the symmetric case.

An algebraic approach, based on consideration of Jacobi matrices and associated quadratic forms; it works only when the equation $F(a, \lambda) = 0$ is algebraic.

These two approaches enabled us to solve the problem in the following cases:

a) When all but 2 of the α_j are integers [21, 22],

b) When three of the α_j are non-integer, and the rest are integers, symmetric case [23],

c) When $n = 4$ and all $\alpha_j = m_j + 1/2$, where $m_j \geq 0$ are integers, symmetric case [17, 18].

We plan to continue this program. First of all, we will generalize b) to the general (non-symmetric) case. The conjectured result is the following:

If $\alpha_0, \alpha_1, \alpha_2$ are not integers, but $\alpha_3, \ldots, \alpha_n$ are integers, then the necessary and sufficient condition of the existence of the metric is

$$\cos^2 \pi\alpha_0 + \cos^2 \pi\alpha_1 + \cos^2 \pi\alpha_2 + 2(-1)^\sigma \cos \pi\alpha_0 \cos \pi\alpha_1 \cos \pi\alpha_2 < 1, \quad (6)$$

where

$$\sigma = \sum_{j=3}^{n} (\alpha_j - 1),$$

and if it is satisfied, there exist $\alpha_3 \ldots \alpha_n$ *non-equivalent metrics, for generic position of the singularities.*

The metrics are called equivalent if the corresponding developing maps are related by $f_1 = L \circ f_2$, where L is a Möbius transformation. This conjectured result is based on a transformation of the hypergeometric equation into an equation (2) with three non-integer exponent differences, which we discovered. The existence of such a transformation was hinted by Klein in his lectures [33, p. 20], in the last paragraph of the 4-th lecture, but Klein does not give a clear statement.

Next we plan to consider the general symmetric case with four singularities, in other words, we wish to determine how many choices of the accessory parameter in the real Heun equation give unitarizable monodromy.

This we plan to do with the geometric method which is based on a complete classification of (geodesic) spherical quadrilaterals.

Classification of spherical quadrilaterals is a part of the problem of classification of all circular quadrilaterals mentioned in section 1. This classification will also permit us to complete the description of real solutions of the Painlevé VI equation in the case of unitary monodromy.

It seems that the algebraic method is limited to the case when there are at most three non-integer angles, so only geometric method remains when there are more. Preliminary results we obtained so far suggest that it will be possible to give a complete classification of spherical quadrilaterals up to isometry.

3. Classification of circular quadrilaterals.

The problems stated in the two previous sections are closely related to classification of circular quadrilaterals. We begin with a formal definition. A *circular polygon* is defined by the following data:

$$(D, t_1, \ldots, t_n, f),$$

where D is a disk in $\overline{\mathbf{C}}$, t_j are distinct points on ∂D, enumerated in the natural order, and $f : \overline{D} \to \overline{\mathbf{C}}$ is a continuous function meromorphic in $\overline{D} \backslash \{t_j\}$ with the following properties:

f is a local homeomorphism on $\overline{D} \backslash \{t_j\}$,

f has a conical singularity at each t_j, that is

$$f(z) = f(t_j) + (c + o(1))(z - t_j)^{\alpha_j}, \quad z \to t_j, \quad \alpha_j > 0, \quad c \neq 0, \qquad (7)$$

and $f([t_j, t_{j+1}])$ is a subset of some circle $C_j \subset \overline{\mathbf{C}}$. Here $[t_j, t_{j+1}]$ is the boundary arc between t_j and t_{j+1}. If $\alpha_j = 0$, the power in (7) must be replaced by the logarithm. The numbers $\pi\alpha_j \geq 0$, are the interior angles of the polygon.

To each circular polygon we associate the *lower configuration*

$$(C_1, a_1, C_2, a_2, \ldots, C_n, a_n),$$

where C_j are the circles containing $f(t_j, t_{j+1})$ and $a_j = f(t_j)$.

The main question is:

How many circular polygons there exist for given angles $\pi\alpha_j$ and given lower configuration consistent with these angles.

If all α_j are integers, f extends by reflection to a rational function, and the problem was completely solved in [15, 20]. This was done by introducing a simple combinatorial invariant which is called the *net*: it is the cell decomposition of D obtained by taking the f-preimage of the lower configuration. In our subsequent work [21, 22, 17, 18] we extended the method to non-integer angles.

The cells of the net are labeled by their f-images. Two polygons are equal if their lower configurations are equal and if their labeled nets are obtained from each other by an orientation-reserving homeomorphism of D. Thus the main question stated above is reduced to a purely combinatorial problem of classification of certain cell decompositions of a disk.

Problems in section 1 require a classification of all circular quadrilaterals. This includes classification of lower configurations, and as a first step, classification of quadruples of circles on the sphere such that $C_j \cap C_{j+1} \neq \emptyset$. It turns out that there are 15 generic types of such quadruples (up to homeomorphisms of the sphere), and the type of the quadruple is a new important topological invariant of a real solution of PVI: it is responsible for the number and order of special points on an interval (see section 1). For example, we conjecture that the number of special points is finite if and only if each two circles of the lower configuration intersect.

Classification of circular quadrilaterals is a difficult problem, and the connection of this problem with Painlevé VI equation which we described in section 1 can be used in two ways: certainly such a classification will shed light on the real solutions of PVI, but it is also possible that modern analytic results on PVI, especially the Jimbo's asymptotics will help with completing the classification.

Problems in section 2 require a classification of *spherical* quadrilaterals (that is, geodesic ones). This means that all C_j are great circles, and there is only one generic configuration of four great circles. This permitted us to obtain a preliminary draft of the classification of spherical quadrilaterals which was mentioned in section 2.

A complete classification of circular triangles was obtained by Klein, with the important applications to the hypergeometric equation. The most comprehensive previous work on classification of circular quadrilaterals is the thesis of Klein's student Ihlenburg [30], who developed the ideas of Klein and Shönflies, and obtained universal relations between the angles and side lengths. But his work falls far short of the complete classification.

4. Entire functions arising as spectral determinants of PT-symmetric boundary value problems.

In this section we consider the eigenvalue problem

$$-y'' + ((-1)^{\ell+1} z^m + \lambda) y = 0, \tag{8}$$

with the boundary conditions $y(z) \to 0$ on two appropriate rays in the complex plane. The values λ for which such a solution exists are called eigenvalues. When $m = 2$, $\ell = 1$, this is equivalent to a harmonic oscillator, which is the only exactly solvable case. When $m = 4$, $\ell = 2$ we have quartic anharmonic oscillators. We notice that anharmonic oscillators with quartic and cubic potentials play a prominent role in the development of quantum mechanics and quantum field theories since the very beginning of these theories. The idea to consider the case of non-integer real α and to study what happens to the eigenvalues when α varies is due to C. Bender, see [1] and references there. Our notation differs from that of Bender by the change of the independent variable $z \mapsto iz$.

It was conjectured by Bessis and Zinn-Justin, and proved by Dorey, Dunning and Tateo [8], that when $m = 3$ and the normalization rays are symmetric with respect to the real line, all eigenvalues are real. To explain this surprising result, Bender proposed to study PT-symmetric boundary value problems. Here PT stands for "parity–time" but for our purposes one can accept the following definition: a problem is called PT-symmetric if the differential operator has real coefficients, while the two normalization rays are symmetric with respect to the real line. When the normalization rays are the positive and negative rays, we obtain Hermitian symmetry, and in this case all eigenvalues are real. For a general PT-symmetric problem, the eigenvalues are symmetric with respect to the real line. If they are real, they say that PT symmetry is unbroken, otherwise it is broken.

To obtain a non-trivial PT-symmetric problem for equation (8) one chooses the two normalization rays

$$\arg z = \pm \frac{\ell + 1}{m + 2} \pi.$$

Bender and Boettcher [1] computed and plotted several smallest real eigenvalues as functions of α for $\ell = 1, 2$ and real m, and observed a remarkable breaking of PT-symmetry.

We propose here to prove these features found in [1] rigorously. So far we have the following preliminary result [13]:

If $m \geq 4$ and $\ell = 2$, then all eigenvalues are real and positive.

This is obtained by a generalization of the method of K. Shin [41], who proved among other things that the spectrum is real and positive when $m \geq 2$ and $\ell = 1$. Our theorem, together with this result of Shin, covers all cases when the spectrum is real according to the computation of Bender and Boettcher.

The main conjecture is the following:

If $m \in (2, 3) \cup (3, 4)$ and $\ell = 2$, then there are only finitely many real eigenvalues, while the arguments of the rest of the eigenvalues accumulate as $\lambda \to \infty$ to the points

$$\pm \frac{4 - m}{2 + m} \pi.$$

The pictures in [1] suggest that for this range of m, ℓ all but finitely many eigenvalues are non-real. Our conjecture about accumulation rays is based on a heuristic argument. We hope to be able to convert this argument to a rigorous one.

Our interest in this problem started with the question of the theory of entire functions asked in [27]: Does there exist an entire function f with all zeros positive and all 1-points (solutions of the equation $f(z) = 1$) lying on two rays different from the positive ray?

In [6] we proved that if such function exists then the two rays where the 1-points lie must be of the form $\arg z = \pm\alpha$. But it is surprising that such entire functions exist at all, and we showed that they indeed exist when $\alpha \in (0, \pi/3)$ and $\alpha = 2\pi/5$. Our examples come from the theory of the differential equation (8). When $m = 3$, the Stokes multiplier of this equation satisfies the functional equation

$$f(\lambda) + f(\omega\lambda)f(\omega^{-1}\lambda) = 1, \quad \omega = 2\pi/5,$$

and as f has all zeros positive, we obtain the example with $\alpha = 2\pi/5$. This functional equation was studied for the first time by Sibuya and Cameron [42]. Recently it was discovered [8, 9] that similar functional equations occur not only in the study of the Stokes phenomenon of (8) but also in the integrable models of statistical mechanics. This analogy with statistical mechanics permitted to prove the reality of the eigenvalues of PT-symmetric problems.

We find this connection of the theory of entire functions with differential equations and integrable models remarkable and deserving further study, and consider the specific problem stated above as a first step.

5. Broader impact of the proposed research.

We expect two kinds of the broader impact:

1. Analytic theory of differential equations is one of the main mathematical tools of physics. Section 4 of this proposal is directly stimulated by questions of physicists [1] and we assume that the proposed research will be useful in PT-symmetric quantum mechanics. The choice of problems in sections 1-3 is motivated by their intrinsic mathematical interest rather than specific applications. However the previous mathematical work of the proposers found various applications, sometimes unexpected, in control theory, material science, computer science, signal processing, physics and astrophysics, and we expect that the proposed research will have similar applications.

2. The proposers expect a contribution of this research on education. They plan to attract graduate, and possibly undergraduate students to participate in this research. The proposer will also continue his efforts in creating public resources which are intended to increase public scientific literacy and public engagement with science and technology. This includes his web pages and active participation in the projects like Math Overflow.

In other words, we expect similar broader impact to our previous NSF funded research.

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