MA440 REAL ANALYSIS (HONORS)

MIDTERM EXAM 1 PRACTICE PROBLEMS

1. Suppose $A \subset \mathbb{R}^p$ and $B \subset \mathbb{R}^q$ and consider the product

$$A \times B = \{(x, y) \in \mathbb{R}^{p+q} : x \in A, y \in B\}.$$

(a) Show that if A and B are open in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, then $A \times B$ is open in $\mathbb{R}^{p+q}$.

(b) Similarly, show that if both A and B are closed, then $A \times B$ is closed.

2. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R}^p$ with the property that there exists a real number r, $0 < r < 1$, and an integer N_0 such that

$$\|x_{n+1} - x_n\| \leq r\|x_n - x_{n-1}\| \quad \text{for all} \quad n \geq N_0.$$

Then prove $(x_n)_{n=1}^{\infty}$ is a convergent sequence.

3. Let $(x_n)_{n=1}^{\infty}$ be a sequence in a compact set $K \subset \mathbb{R}^p$ that is *not* convergent. Show that there are two subsequences of this sequence that are convergent to *different* limit points.

4. We say that x is a *limit point* of the sequence (x_n) if $x = \lim_{k\to\infty} x_{n_k}$ for a certain subsequence (x_{n_k}). Construct a compact set of real numbers whose limit points form a countable set.

5. Prove that if A and B are disjoint closed sets in $\mathbb{R}^n$, then $A \cup B$ is disconnected.

6. Let (a_n) be a sequence of real numbers of such that $a = \lim_{n\to\infty} a_n$. Show that the sequence of arithmetic means

$$\alpha_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

is also convergent and $\lim_{n\to\infty} \alpha_n = a$.

7. Find the upper and lower limits of the sequence (x_n) defined by

$$x_1 = 0, \quad x_{2m} = \frac{x_{2m-1}}{2}, \quad x_{2m+1} = \frac{1}{2} + x_{2m}.$$

8. Suppose that (x_n) and (y_n) are Cauchy sequences in $\mathbb{R}^p$. Show that the numeric sequence $d_n = \|x_n - y_n\|$ is convergent.