MA440 REAL ANALYSIS (HONORS)

MIDTERM EXAM 2 PRACTICE PROBLEMS

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that $\lim_{|x|\to\infty} f(x) = 0$, i.e., for all $\epsilon > 0$, there is an N so that $|f(x)| < \epsilon$ form all x with $|x| > N$. Show that f is uniformly continuous on $\mathbb{R}^n$.

2. Let f be a continuous function on $\mathbb{R}$ to $\mathbb{R}$ which does not take on any of its values twice. It it true that f must either be strictly increasing (in the sense that if $x' < x''$ then $f(x') < f(x'')$) or strictly decreasing?

3. Let f be defined for all real x, and suppose that $|f(x) - f(y)| \leq (x-y)^2$ for all real x and y. Prove that f is constant.

Hint: Show that f is differentiable and find f'.

4. Let F be the Cantor set. Let f be a bounded real function on $[0,1]$ which is continuous at every point outside F. Prove that f is Riemann integrable on $[0,1]$.

Hint: F can be covered by finitely many segments whose total length can be made as small as desired.

5. If $f : [0,1] \to [0,\infty)$ is increasing and $f\left(\frac{1}{2}\right) > 1$, show that

$$\int_0^1 f(x)dx > \frac{1}{2}.$$

6. Show that $f_n(x) = n\sin(x/n)$ converges uniformly on $[-a,a]$ for any finite $a > 0$ but does not converge uniformly on $\mathbb{R}$.

7. Show that a monotone function $f : [a,b] \to \mathbb{R}$ is continuous if and only if its image $f([a,b])$ is an interval.

8. Suppose

- f is continuous for $x \geq 0$,
- $f'(x)$ exists for $x > 0$,
- $f(0) = 0$,
- f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x}, \quad x > 0$$

and prove that g is monotonically increasing.