Small oscillations

A. Eremenko

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To write equations of motion of a system of points, one can use the Newton law of motion $F = ma$. This is not always convenient for systems with restraints because one has to take reaction forces into account which is not always easy.

In the more general and flexible approach proposed by Hamilton one describes the state of the system by its generalized coordinates $q_1, \ldots, q_n$ and generalised velocities $\dot{q}_k = dq_k/dt$. All one needs is an expression of kinetic energy E and potential energy U in terms of q_k and $\dot{q}_k$. Then one forms the function $L = E - U$ which is called the Lagrangian, and the equations of motion have the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_j}. \tag{1}$$

For example , for a particle of mass m moving in a field with potential $U(q)$ we have

$$L(\dot{q}, q) = m\dot{q}^2/2 - U(q)$$

and the equation of motion is

$$\frac{d}{dt}(m\dot{q}) = -\frac{\partial U}{\partial q}.$$

The RHS is of course the force, and we obtain the usual Newton's equation.

The advantage of this method is seen when we consider the systems with constraints. As an example we take the double pendulum which consists of two masses connected by two rods as shown in the figure. Let q_1 and q_2 be the angles mde by the rods with the vertial direction. Then kinetic energy of the first mass is

$$\frac{m_1}{2}\ell_2^2\dot{q}_1^2,$$

and kinetic energy of the second mass is

$$\frac{m_2}{2}\left(\ell_1^2\dot{q}_1^2+\ell_1^2\dot{q}_2^2+2\ell_1\ell_2\cos(q_2-q_1)\dot{q}_1\dot{q}_2\right).$$

Potential energy of the system is

$$U=-(m_1+m_2)g\ell_1\cos\dot{q}_1-m_2g\ell_2\cos\phi_2.$$

Thus

$$\begin{aligned} L &= \frac{m_1+m_2}{2}\ell_1\dot{q}_1^2+\frac{m_2}{2}\ell_2^2\dot{q}_2^2+m_2\ell_1\ell_2\cos(q_2-q_1)\dot{q}_1\dot{q}_2 \\ &+ (m_1+m_2)g\ell_1\cos\dot{q}_1-m_2g\ell_2\cos\phi_2. \end{aligned}$$

The equations of motion are non-linear and difficult to solve, so we *linearlize* the system, considering the motion near the equilibrium $q_1=q_2=0$. This means that all cosines are replaced by 1, and we have the Lagrangian of the linearized system