Geometric theory of meromorphic functions

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Dedicated to A. A. Goldberg.

ABSTRACT. This is a survey of results on the following problem. Let X be a simply connected Riemann surface spread over the Riemann sphere. How are the properties of the uniformizing function of this surface related to the geometric properties of the surface? Based on the lectures in U. Michigan in May 2006.

1. According to the Uniformization Theorem, for every simply connected Riemann surface X there exists a conformal homeomorphism $\phi : X_0 \to X$, where X_0 is one of the three standard regions, the Riemann sphere $\overline{\mathbf{C}}$, the complex plane $\mathbf{C}$ or the unit disc $\mathbf{U}$. We say that the *conformal type* of X is *elliptic, parabolic* or *hyperbolic*, respectively. The map ϕ is called the *uniformizing map.* If X is given by some geometric construction, the problem arises to relate properties of ϕ to those of X. This includes the determination of the conformal type of X [**14, 41, 52, 53**]. The case which was studied most is that $X \subset \mathbf{C}$ is a simply connected region, $X \neq \mathbf{C}$. Then X is of hyperbolic type and ϕ is a univalent function in $\mathbf{U}$. An example of the result relating geometric properties of $X \subset \mathbf{C}$ and properties of ϕ is the classical theorem of Caratheodory: ϕ is continuous in the closed disc if and only if ∂X is locally connected. Another example is the Ahlfors distortion theorem which relates the growth of a uniformizing map to the geometry of the image domain.

We recall a more general construction of X. A surface *spread over the sphere* is a pair (X, p), where X is a topological surface and $p : X \to \overline{\mathbf{C}}$ a continuous, open and discrete map. This map p is usually called the *projection.* The natural equivalence relation is $(X, p) \sim (Y, q)$ if there is a homeomorphism $\phi : X \to Y$ with the property $p = q \circ \phi$. According to a theorem of Stoïlov, every continuous open and discrete map p between surfaces locally looks like $z \mapsto z^n$. Those points where $n > 1$ are isolated, they are called *critical points*, or multiple points of multiplicity n. Stoïlov's theorem implies that there is a unique conformal structure on X which makes p holomorphic. If ϕ is a uniformizing map, then $f = p \circ \phi$ is a meromorphic function in one of the three standard regions $\overline{\mathbf{C}}, \mathbf{C}$ or $\mathbf{U}$. The surface (X, p) spread over the sphere is then the "Riemann surface of f^{-1}". We also call it the surface associated to f, or say that f is associated with (X, p).

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If D is a region on the sphere, a *branch* of p^{-1} in D is a continuous function $\psi : D \to X$ such that $p \circ \psi = \mathrm{id}_D$.

We can define the length of a curve in X as the spherical length[1] of its image under p. Then X becomes a metric space with an *intrinsic metric*, which means that the distance between two points is the infimum of the lengths of curves connecting these points. Similarly, if $p : X \to \mathbf{C}$, and the Euclidean metric in $\mathbf{C}$ is used to measure lengths of curves, we obtain a Riemann surface *spread over the plane*.

The intrinsic metric on X is a smooth Riemannian metric of constant curvature on the complement of the critical set of p. It is easy to show that the intrinsic metric on X determines the projection p up to an isometry of the image sphere or the plane. In what follows, unless otherwise stated, (X, p) denotes a simply connected surface spread over the sphere, equipped with the intrinsic spherical metric.

Some criteria of conformal type can be stated in terms of topological (or even set-theoretic) properties of p. For example, Picard's theorem implies that X is of hyperbolic type if p omits three points.

The following result is due to Nevanlinna. A point $a \in \overline{\mathbf{C}}$ is called a *totally ramified value* (of multiplicity $m \geq 2$) of a surface (X, p) spread over the sphere, if all preimages of a under p are multiple, (of multiplicity at least m). We allow $m = \infty$ which means that the value a is omitted.

1.1 NEVANLINNA'S THEOREM *If a surface spread over the sphere has q totally ramified values of multiplicity m_k, $1 \leq k \leq q$, and*

$$\sum_{k=1}^{q} \left(1 - \frac{1}{m_k}\right) > 2,$$

then the surface is hyperbolic. In particular, a parabolic surface has at most four totally ramified values.

The example $(\mathbf{C}, \wp)$ shows that a parabolic surface can indeed have 4 totally ramified values. Moreover, one can easily show that the equation

$$\sum_{k=1}^{q} \left(1 - \frac{1}{m_k}\right) = 2$$

has 5 solutions up to a permutation of the m_k, and to each of these solutions corresponds an elliptic or trigonometric function. A very short proof of Theorem 1.1 can be found in [**46**].

1.2 AHLFORS'S FIVE ISLANDS THEOREM. *Suppose that for five Jordan regions with disjoint closures on the sphere, there are no branches of p^{-1} in any of these regions. Then X is of hyperbolic type.*

This theorem was stated for the first time by Bloch [**9**] (with discs instead of Jordan regions) and proved by Ahlfors in [**2**], as a corollary from his "Uberlagerungsflachentheorie". It is a recent discovery [**4**] that actually Theorem 1.2 can be derived from Theorem 1.1 by a simple argument. This derivation is based on the following important principle:

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[1] We choose the spherical length element to be $2|dz|/(1 + |z|^2)$, so that the curvature of the spherical metric is $+1$.

1.3 Zalcman's Lemma. *Let F be a family of meromorphic functions in some region D, which is not normal in D. Then there exists a sequence $f_n \in F$, and two sequences $r_n > 0$ and $z_n \in D$ such that there exists a non-constant limit*

$$f(z) = \lim_{n\to\infty} f_n(r_n z + z_n),$$

uniform on compact subsets of $\mathbf{C}$, *and moreover,*

$$f^{\#}(z) := \frac{|f'(z)|}{1+|f(z)|^2} \leq f^{\#}(0) = 1, \quad z \in \mathbf{C}.$$

Proof. Without loss of generality we may assume that D is the unit disc, functions of the family F are meromorphic in the closure of D, and F is not normal at 0. This means that there exists sequence $f_n \in F$ and $w_n \to 0$ such that $f_n^{\#}(w_n) \to \infty$. Then

$$\max_{z\in D}(1-|z|)f_n^{\#}(z) = (1-|z_n|)f_n^{\#}(z_n) \geq (1-|w_n|)f_n^{\#}(w_n) \to \infty.$$

Thus $r_n = 1/f_n^{\#}(z_n) = o(1-|z_n|)$. We claim that r_n and z_n have the required property. Indeed, putting $g_n(z) = f_n(r_n z + z_n)$, we obtain $g_n^{\#}(0) = 1$ and

$$g_n^{\#}(z) = r_n f_n^{\#}(r_n z + z_n) \leq r_n f_n^{\#}(z_n)\frac{1-|z_n|}{1-|r_n z + z_n|} \leq \frac{1-|z_n|}{1-|z_n|-r_n|z|} \to 1,$$

because $r_n = o(1-|z_n|)$. So g_n is a normal family. After selecting a subsequence we get $g_n \to f$ for some meromorphic function f, $f^{\#}(z) \leq 1, z \in \mathbf{C}$ and $f^{\#}(0) = 1$, which proves all statements of the lemma.

Now we derive Theorem 1.2 by contradiction. Suppose that f is a meromorphic function in the plane, and D_k are five disjoint Jordan regions such that there are no inverse branches of f^{-1} in D_k, $1 \leq k \leq 5$. Choose five points a_k on the sphere and consider quasiconformal homeomorphisms ψ_n of the sphere which map each D_k into $1/n$-neighborhood of a_k. Then the surfaces $(\mathbf{C}, \psi_n \circ f)$ spread over the sphere are all parabolic because the maps $\psi_n \circ f$ are *quasiregular*[2], so there exist homeomorphisms ϕ_n of $\mathbf{C}$ such that $f_n = \psi_n \circ f \circ \phi_n$ are meromorphic functions. These functions have no inverse branches over $1/n$ neighborhoods of a_k. We can use arbitrarity in the choice of ϕ_n to normalize our functions: $f_n(0) = 0$, $f_n'(0) = 1$. If the family $\{f_n\}$ is normal in the whole plane, then the limit functions are non-constant because of the normalization, and it is easy to see that a_k are totally ramified values of these functions, contradicting Theorem 1.1. If $\{f_n\}$ is not a normal family in the plane, we apply Zalcman's lemma to make it normal, and again obtain a contradiction.

Zalcman's lemma shows the importance of study of meromorphic functions with bounded spherical derivative. Very little is known about this class. We mention a theorem of Clunie and Hayman that if an entire function has bounded spherical derivative than it is at most of order one, normal type. (A typical meromorphic function of this class has order two, normal type).

Sullivan asked the general question, for which surfaces (X, p) the conformal type is determined by topological properties of p. More precisely, let us say that a simply connected surface (X, p) spread over the sphere has a *stable type* if for every homeomorphism $\psi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ the surface $(X, \psi \circ p)$ has the same type as

[2]Quasiregular maps in dimension 2 are compositions of holomorphic functions with quasiconformal maps. Quasiconformal maps preserve the conformal type of a simply connected Riemann surface.

(X,p). For example, surfaces satisfying the conditions of Theorem 1.1 or 1.2 are of stable hyperbolic type. Another interesting class of surfaces of stable type is the *Speiser class* S. We say that $(X,p) \in S$ if there exists a finite set $A \subset \overline{\mathbf{C}}$ such that the restriction $p : X \backslash p^{-1}(A) \to \overline{\mathbf{C}} \backslash A$ is a covering. The stability of type of such surfaces was proved by Teichmüller in [**49**] as one of the first applications of quasiconformal mappings. The argument is based on the following simple

LEMMA. *Let f be a function of Speiser class, in $\{z : |z| < R\}$, where $R = 1$ or $R = \infty$, and A is the finite set as above. Let $f_1 = \psi \circ f \circ \phi$, where f_1 is a meromorphic function and ψ and ϕ are homeomorphisms, ϕ leaves 0 and $1/2$ fixed. Suppose that there is an isotopy $\psi_t, 0 \leq t \leq 1$, $\psi_0 = \mathrm{id}$, $\psi_1 = \psi$ such that the elements of $A \cup \{f(0), f(1/2)\}$ remain fixed by each ψ_t. Then $f_1 = f$.*

By the covering homotopy theorem there exists an isotopy ϕ_t, $\phi_1 = \phi$ such that $\psi_t \circ f \circ \phi_t = f_1$, $0 \leq t \leq 1$. The functions $\phi_t(0)$ and $\phi_t(1/2)$ are continuous and can take discrete sets of values, hence $\phi_t(t)(0) = 0$ and $\phi_t(1/2) = 1/2$ for all t. Putting $t = 0$ we obtain $f \circ \phi_0 = f_1$. It is easy to conclude from the last equality that ϕ_0 is a conformal homeomorphism of $\{z : |z| < R\}$. On the other hand, it fixes 0 and 1/2, so $\phi_0 = \mathrm{id}$ and $f_1 = f$.

Now Teichmüller's result follows because every homeomorphism is isotopic to a quasiconformal homeomorphism modulo any given finite set, and quasiconformal homeomorphisms preserve conformal type of a simply connected surface.

Teichmüller's argument extends to the somewhat wider class consisting of surfaces with the property that the distances between their singularities[3] are bounded from below by a positive constant. All surfaces of stable parabolic type known to the author have this property. The simplest example of a parabolic surface with a non-isolated singularity is $(\mathbf{C}, p)$, with $p(z) = \sin z/z$, and it follows from a result of Volkovyskii [**52**, Th. 45] that the conformal type of this surface is not stable.

2. If we take the five regions in Ahlfors's theorem to be spherical discs of equal radii, Theorem 1.1 implies the following [**1**]: *Suppose that for some $\epsilon > 0$ there are no branches of p^{-1} in discs of radii $\pi/4 - \epsilon$. Then X is of hyperbolic type.* The question arises, what is the best constant for which this result still holds. Let $B(X,p)$ be the supremum of radii of discs where branches of p^{-1} exist, and $B = \inf B(X,p)$, where the infimum is taken over all surfaces of elliptic or parabolic type. Ahlfors's estimate $B \geq \pi/4$ was improved by Pommerenke [**43**] to $B \geq \pi/3$, and recently the sharp result was obtained in [**11**]:

$$B = b_0 := \arccos(1/3) \approx 0.39\pi.$$

We have $B(\mathbf{C}, \wp) = B$, where $\wp$ is the Weierstrass function of a hexagonal lattice. It is interesting to notice that $B = b_0$ implies Theorem 1.1 by a simple argument given in [**11**].

For surfaces (X,p) of elliptic type we have $B(X,p) > b_0$, but it is not known whether the constant b_0 is best possible in this inequality.

We sketch the proof for elliptic surfaces. First one constructs a triangulation T of X into geodesic triangles, so that the vertices of this triangulation coincide with the set of critical points, and the circumscribed radius of each triangle is at most $B(X,p)$. This is always possible to do if $B < \pi/2$ which we can assume. Suppose

[3] A formal definition of singularities and distance between them is in section 3.

now that $B(X,p) \leq b_0$. Then the circumscribed radius of each triangle is at most b_0, and an elementary geometric argument shows that the area of each triangle is at most π. Notice that by Gauss formula, area$\,(\Delta) = \sum \alpha(\Delta) - \pi$, where $\alpha(\Delta)$ is the sum of the angles of Δ. As area$\,(\Delta) \leq \pi$ we conclude that area$\,(\Delta) \leq \alpha(\Delta)/2$. If we denote by $\alpha(v)$ the total angle at a vertex, then $\alpha(v) = 4\pi$, assuming that all critical points have multiplicity 2. If d is the degree of our rational function then the total area is

$$4\pi n = \sum_{\Delta \in T} \text{area}\,(\Delta) \leq \frac{1}{2} \sum_{v} \alpha(v) = 2\pi(2n-2),$$

and this is a contradiction.

The proof of $B \geq b_0$ for parabolic surfaces is more complicated. For our class of surfaces with intrinsic metric, one can define integral curvature [**45**] as a signed Borel measure on X which is equal to the area on the smooth part of X and has negative atoms at the critical points of p. The assumption that $B(X,p) < b_0$ implies that the atoms of negative curvature are sufficiently dense on the surface, so that on large pieces of X the negative part of the curvature dominates the positive part. Then a bi-Lipschitz modification of the surface is made, which spreads the integral curvature more evenly on the surface, resulting in a surface whose Gaussian curvature is bounded from above by a negative constant, and the Ahlfors–Schwarz lemma implies hyperbolicity. A non-technical exposition of the ideas of this proof is given in the survey [**10**] which contains some further geometric applications of this technique of spreading the curvature by bi-Lipschitz modifications of a surface.

3. To formalize the notion of a singular point of a multi-valued analytic function, Mazurkiewicz [**39**] introduced another metric, $\rho(x,y) = \inf\{\operatorname{diam} p(C)\}$ on X, where diam is the diameter with respect to the spherical metric and the infimum is taken over all curves $C \subset X$ connecting x and y. Every point $x \in X$ has a neighborhood where the Mazurkiewicz metric coincides with the intrinsic one, but in general the Mazurkiewicz metric is smaller. For example, on the surface $(\mathbf{C}, \cos)$ spread over the sphere, the intrinsic distance between 0 and $2\pi k$ is $2\pi k$, while the Mazurkiewicz distance is π.

Let X^* be the completion of X with respect to the Mazurkiewicz metric. Then p has a unique continuous extension to X^*. The elements of the set $Z = X^* \backslash X$ are called *transcendental singularities* of (X,p). The simplest example of a transcendental singularity is a logarithmic branch point. The *algebraic singularities* are just the critical points of p. The images of singularities under p will be called *singular values.* The images of critical points are called *critical values.*

To each transcendental singularity corresponds an *asymptotic curve* $\gamma : [0,1) \to X$ which has no limit in X but its image $p \circ \gamma$ has a limit in $\overline{\mathbf{C}}$. This limit is called an *asymptotic value* and it is the projection of the singularity.

If $D \subset \overline{\mathbf{C}}$ is a region containing no singular values then the restriction $f : f^{-1}(D) \to D$ is a covering map. The closure of the set of singular values is characterized by this property. However the set of singular values need not be closed.

If X is of parabolic type, then the set of singularities is totally disconnected. This can be proved by using Iversen's theorem [**33, 41**]. The following classical result [**41**], which implies Iversen's theorem, shows that if we "look in all directions from a point" on a parabolic surface then very few singularities are visible.

3.1 Gross's Theorem. *Let (X, p) be a simply connected surface of parabolic type spread over the sphere, and $x \in X$. Then X contains a geodesic ray from x of length π (that is from x to the "antipodal point") in almost every direction.*

It is not known whether the estimate of the size of the exceptional set in this theorem can be improved, but there are examples where this exceptional set of directions has the power of the continuum [**52**, Th. 17]. We include a sketch of such example. Let D be the unit disc, and $E \subset \partial D$ a Cantor set whose complement consists of the arcs $L_k = (a_k, b_k)$. We assume that ∂D has the standard orientation which induces orientation from a_k to b_k on these arcs. Let us cut the surface $(\mathbf{C}, (a_k - b_k e^z)/(1 - e^z))$ along a simple arc that projects to to L_k and call X_k the piece which lies on the right of the arc. Such surfaces X_k are called *logarithmic ends*. Now, for each k we paste X_k to D along the arc L_k, respecting the projection. The result is an open simply connected surface X spread over the sphere. It is easy to show, that this surface can be parabolic if the complementary arcs to $\cup_{k=1}^{n} L_k$ decrease sufficiently fast as $n \to \infty$. To do this one approximates X by surfaces with finitely many singularities which are all parabolic by Theorem 4.2 below. Volkovyski obtained a quantitative sufficient condition for X to be of parabolic type, but it is much stronger than the only known necessary condition that E is of zero logarithmic capacity. Evidently, on X, the length of a geodesic ray from the center of D in the direction of any point of E is $\pi/2$.

The projection of the set Z of singular points is an analytic (Suslin) set [**39**], and for every analytic subset A of the sphere one can find surfaces of both parabolic and hyperbolic types for which $A = p(Z)$ [**30**].

The following classification of transcendental singularities was introduced by Iversen [**33**]. A singular point $x \in X^* \backslash X$ is called *direct* if for some neighborhood $V \subset X^*$ of x, the map p omits $p(x)$ in $V \backslash \{x\}$. Otherwise x is called *indirect*. For example, $(\mathbf{C}, \sin z/z)$ has two direct singularities over ∞ and two indirect singularities over 0. The following result was proved in [**31**]:

3.2 Heins's Theorem. *For a parabolic Riemann surface spread over the sphere, the set of projections of direct singularities is at most countable.*

On the other hand, for some parabolic surfaces, the set of direct singularities lying over one point may have the power of the continuum. To construct such example, one can take an infinite dyadic tree T properly embedded in the plane, and consider a simply connected neighborhood U of T, so small that all infinite branches of T define different accessible boundary points of U at infinity. Then one constructs an entire function f that tends to infinity along each infinite branch of T but remains bounded in the complement of U. Then the surface $(\mathbf{C}, f)$ has uncountably many singularities over ∞, all of them direct, because ∞ is omitted. An explicit construction of such entire function f is given in [**7**].

To state further results on direct singularities we recall the notion of the order of a meromorphic function in the plane. Let (X, p) be an open simply connected Riemann surface of parabolic type, spread over the sphere. Then the intrinsic metric defines a notion of area on X. If ϕ is a uniformizing map and $f = p \circ \phi$ the corresponding meromorphic function in $\mathbf{C}$, then the "average covering number" of the sphere by the images of the discs $D(r) = \{z : |z| \leq r\}$ is defined as the area of $\phi(D(r))$ divided by the area of the sphere $\overline{\mathbf{C}}$, which is the same as

$$A(r, f) = \frac{1}{\pi} \int_{D(r)} \frac{|f'|^2}{(1 + |f|^2)^2}, \quad r \geq 0,$$

and the order of f is defined as

$$\lambda(f) = \limsup_{r \to \infty} \frac{\log A(r, f)}{\log r}.$$

It is easy to verify that the order depends only on (X, p) rather then on the choice of the uniformizing map ϕ.

3.3 Denjoy–Carleman–Ahlfors Theorem. *If f is a meromorphic function in the plane, and the Riemann surface $(\mathbf{C}, f)$ has $k \geq 2$ direct singularities, then*

$$\liminf_{r \to \infty} r^{-k/2} A(r, f) > 0.$$

If p omits a point $a \in \overline{\mathbf{C}}$, then a theorem of Lindelöf implies that at least half of all singularities of (X, p) lie over a, and these singularities are evidently direct. So we obtain that such surface can have at most $2\lambda(f)$ singularities over the points in $\overline{\mathbf{C}} \backslash \{a\}$. As a corollary, an entire function of order λ can have at most 2λ finite asymptotic values. This is in contrast with the case of meromorphic functions: there exist meromorphic functions in the plane of arbitrary prescribed order $\lambda \geq 0$ for which every point on the sphere is an asymptotic value [**18**].

In the simplest example $\sin z/z$ mentioned above, the indirect singularities over 0 are accumulation points of critical points. A result of Volkovyski [**52**, Th. 17] shows that this is not always so: there are parabolic surfaces without critical points, having indirect singularities. However, for functions of finite order, the following theorem was proved in [**5**]:

3.4 Theorem. *Let f be a meromorphic function of finite order, and let a be an indirect singularity of $(\mathbf{C}, f)$. Then every neighborhood of a contains critical points z such that $f(z) \neq f(a)$.*

This theorem can be used to prove the existence of critical points under certain circumstances, and more generally, to study the value distribution of derivatives. In [**5**] it was used to prove a result which was conjectured by Hayman: for every transcendental meromorphic function f in $\mathbf{C}$, the equation $ff' = c$ has infinitely many solutions for every $c \in \mathbf{C} \backslash \{0\}$. There is no growth restriction on f in this last result.

The scheme of the proof of Hayman's conjecture is the following. Let $g(z) = z - f^2/(2c)$, and suppose first that f is of finite order. Then each zero of f is a fixed point of g with multiplier 1. According to Fatou's theorem from holomorphic dynamics, to each such fixed point a region of immediate attraction is attached, and this region contains a singular value of g. By Theorem 3.4, g has to have infinitely many critical values. And each critical point of g is a solution of $ff'(z) = c$. The case that f is of finite order and has finitely many zeros is treated separately by elementary considerations.

To extend the result to functions of infinite order one uses the following generalization of Zalcman's lemma which is due to X. Pang: *Let F be a non-normal* *family of meromorphic functions and* $-1 < k < 1$*. Then there exist sequences* $f_n \in F$, $r_n > 0$ *and* z_n *such that the limit*

$$\lim r_n^{-k} f_n(r_n z + z_n) = g$$

exists on every compact in the plane, and g *is a non-constant meromorphic function with bounded spherical derivative (so the order of* g *is at most* 2*).*

To complete the proof of Hayman's conjecture one uses $k = 1/2$.

Goldberg and Heins independently noticed that in Theorem 3.3 one may count some indirect singularities, so-called K-singularities, together with the direct ones. However a geometric characterization of K-singularities is known only for a very special class of symmetric surfaces [**26**].

4. There are few instances when a precise correspondence can be established between classes of surfaces spread over the sphere and classes of meromorphic functions corresponding to them. We mention first a class of hyperbolic surfaces, spread over the plane.

4.1 THEOREM. *For a surface* (X, p) *spread over the plane, the following conditions are equivalent:*
(a) The (Euclidean) radii of discs where branches of p^{-1} *exist are bounded.*
(b) A linear isoperimetric inequality holds on X*.*
(c) X *is of hyperbolic type and the uniformizing map* $\phi : \mathbf{U} \to X$ *is uniformly continuous with respect to the hyperbolic metric on* $\mathbf{U}$ *and the intrinsic (Euclidean) metric on* X*.*

The equivalence between (a) and (c) is essentially Bloch's theorem [**8**]. For the equivalence of (b) to the other two conditions the reference is [**12**] where Theorem 4.1 is stated for a more general classes of surfaces with intrinsic metric (not necessarily spread over the plane). Holomorphic functions $f = p \circ \phi$, where ϕ satisfies (c), are called Bloch functions. This class is important because of its connection with univalent functions: if g is univalent in $\mathbf{U}$, then $\log g'$ is a Bloch function, and every Bloch function has the form $c \log g'$ where g is univalent in $\mathbf{U}$ and c is a constant [**43**]. In the case of meromorphic functions and surfaces spread over the sphere, Ahlfors and Dufresnoy [**15**] proved that (b) implies (c). It is plausible that (b) is actually equivalent to (c)[4], but it is not clear what sort of conditions could replace (a) in the spherical case. Functions satisfying (c) with the spherical metric instead of the Euclidean one are called *normal*, and they were much studied, see, for example [**44**].

Passing to parabolic surfaces, we first describe a way to visualize certain classes of surfaces spread over the sphere, in particular, the Speiser class defined in section 1. Suppose that all singular values belong to some Jordan curve γ on the sphere. We call such surfaces *cellular surfaces* and the corresponding meromorphic functions *cellular functions*.

The full preimage $p^{-1}(\gamma)$ defines a cell decomposition of X. We recall that a cell decomposition of a surface X is a representation of X as a locally finite union of disjoint sets called *cells*. Each cell is a homeomorphic image of a point (vertex), or of an open interval (edge) or of an open disc (face). The closure of each cell is a union of this cell with cells of smaller dimension. Notice that the boundary of a

[4]This was proved by Bruce Kleiner (unpublished).

face can contain infinitely many edges or infinitely many vertices. The 1-*skeleton* of a cell decomposition is the union of edges and vertices.

Now consider a cellular surface (X, p) spread over the sphere, so that all singular values belong to a Jordan curve γ. One can show that the full preimage of γ is a 1-skeleton of a cell decomposition of X. Its vertices of degree $2m$ are the multiple points of multiplicity m of p. One can add vertices of degree 2 by breaking an edge into two edges, which is sometimes convenient. The curve γ divides the sphere into two regions, D_0 and D_2, which contain no singular values. It follows that each face is mapped on D_1 or D_2 homeomorphically. We can assign colors to the faces: those mapped on D_0 are white and those mapped on D_1 are black. Then our cell decomposition has the property that whenever two faces have a common boundary edge, they are of different color. This is consistent with the property that all vertices have even degrees.

If (X, p) is of the Speiser class, we have a finite set A that contains all singular values. Let γ be a Jordan curve in $\mathbf{C}$ passing through all points of the set A. Consider the cell decomposition of the sphere whose 2-cells are the two complementary domains to γ, 0-cells are the points of A, and 1-cells are the arcs of γ between these points. The preimage of this cell decomposition under p is a cell decomposition of X. It essentially coincides with the cell decomposition described above, except that now we defined precisely which points are vertices of multiplicity 2, these are exactly the simple preimages of the elements of the set A.

In classical literature they usually consider the dual graph of this cell decomposition of X, called the *line complex*, which can be obtained in the following way. Let e_k be the edges of γ between adjacent pairs of the points of A. Choose two point x and o, one in each complementary region of γ and connect these points by simple disjoint arcs γ_k, so that γ_k intersects γ only once, and this intersection occurs at a point of e_k. The points x and o and the arcs γ_k form a graph with two vertices and $q = \operatorname{card} A$ edges. The preimage of this graph under p is a properly embedded connected graph in X which is called the line complex of (X, p). Every vertex of the line complex has degree q and every edge connects two vertices of different type, one projecting to x another to o. All possible line complexes are characterized by these two properties. The components of the complement of a line complex are called faces. There can be faces with an even number $2n$ of boundary edges, they correspond to critical points of order n; and there can be unbounded faces with infinitely many boundary edges, they correspond to the "logarithmic branch points" of the surface (X, g). The logarithmic branch points constitute the simplest example of transcendental singularities.

Teichmüller's theorem mentioned in Section 1 implies that the conformal type of (X, g) is determined by its line complex.

The following question was asked by A. Epstein. Let $(\mathbf{C}, f)$ be a parabolic surface of Speiser class, such that f is of finite order. If ψ is a homeomorphism of the Riemann sphere, then $(\mathbf{C}, \psi \circ f)$ is parabolic because as we mentioned before, Speiser surfaces have stable type. Let ϕ be the uniformizing map of this deformed surface, so that $g = \psi \circ f \circ \phi$ is a meromorphic function in the plane. It is easy to show that g also has finite order. The question is whether the orders of f and g are the same. In other words, does the line complex determine the order of a function? Künzi [**36**] showed that this is not necessarily so for meromorphic functions f, but the question remains unsolved for entire functions.

Next theorem gives a complete characterization of meromorphic functions whose inverses have finitely many singularities.

4.2 THEOREM. *For a surface (X, p) spread over the sphere, the following conditions are equivalent:*
(a) The set of transcendental singularities of (X, p) is finite, and p has finitely many critical points.
(b) (X, p) is equivalent to $(\mathbf{C}, f)$ where f is a meromorphic solution of the differential equation

$$f'''/f' - (3/2)\left(f''/f'\right)^2 = P, \tag{0.1}$$

where P is a rational function.

If p has no critical points then P is a polynomial, and for every polynomial P all solutions of (0.1) *are meromorphic functions in the plane. The degree of P is $n-2$, where n is the number of transcendental singularities of (X, g).*

The case when P is a polynomial and p has no critical points is due to R. Nevanlinna [**42**], and the generalization with finitely many critical points to his student Elfving [**17**]. Every solution f of the Schwarz differential equation in (b) is a ratio of two linearly independent solutions of the linear differential equation

$$w'' + (P/2)w = 0. \tag{0.2}$$

This provides very precise information on the asymptotic behavior of f sufficient to prove that (b) implies (a).

We give a sketch of Nevanlinna's argument, assuming for simplicity that there are no critical points. To show that (a) implies (b), Nevanlinna uses the line complex of (X, p) to construct a sequence of rational functions (f_m) such that the surfaces (D_m, f_m), where $D_1 \subset D_2 \subset \ldots \to \mathbf{C}$ form an exhaustion of the plane by some simply connected regions. These surfaces (D_m, f_m) approximate (X, p) in the following sense. There is an exhaustion $X_1 \subset X_2 \subset \ldots \to X$ by open topological discs, such that each X_m is isometric to a subset of the surface (D_m, f_m). The rational functions f_m have critical points of high multiplicity, one for each transcendental singularity of (X, p). So the number of critical points of f_m is bounded independently of m. The rational functions f_m are normalized by $f_m(0) = 0$ and $f'_m(0) = 1$. It follows from the Caratheodory Convergence theorem that the sequence (f_m) converges in some neighborhood of 0 to a holomorphic function f. The crucial observation is that the Schwarzian derivatives

$$P_m = f'''_m/f'_m - (3/2)(f''_m/f'_m)^2$$

are rational functions whose degrees are bounded independently of m. This is because the poles of a Schwarzian derivative can occur only at the critical points of the function, and all these poles are of multiplicity two. As the P_m converge in a neighborhood of 0, they have to converge in the whole plane to a rational function P. The limit function f satisfies the differential equation (0.1), and has no critical points. So P is a polynomial, and f has an analytic continuation to a meromorphic function in the plane as a solution of the differential equation (0.1). Now it is easy to see that f is the meromorphic function associated with (X, p).

To show that that (b) implies (a) we first notice that f has no critical points (each critical point in $\mathbf{C}$ is a pole of the Schwarzian derivative, and we assume that P is a polynomial). Then one writes $f = w_1/w_0$ where w_1 are entire functions, linearly independent solutions of the differential equation (0.2). Asymptotic integration of this linear differential equation shows that the surface associated to f has finitely many singularities, which proves (a).

The singularities of $(\mathbf{C}, f)$ in Theorem 4.2 are simply related to the so-called Stokes multipliers of the equation (0.2). As it is easy to construct a surface as in (a) with prescribed projections of singularities, one can derive the existence of a linear differential equation (0.2) with prescribed Stoke's multipliers [**48**].

In the case of infinitely many algebraic singularities, one cannot obtain such complete results, but still for many subclasses of surfaces of the Speiser class (which was defined in Section 1) one can obtain rather complete information about the asymptotic behavior of the uniformizing functions [**13, 25, 27, 36, 53**].

5. Our next example of an exact correspondence between a class of functions and a class of surfaces involves a class of cellular surfaces spread over the plane and having a symmetry property. Suppose that an anticonformal involution $s : X \to X$ is given. The set of fixed points of s will be called the axis. We say that (X, p) is *symmetric* if $p \circ s = \overline{p}$ where the bar stands for the complex conjugation. A symmetric surface spread over the plane is called a *MacLane surface* if all its critical points belong to the axis, and for each transcendental singularity over $\mathbf{C}$, the axis can serve as an asymptotic curve. Evidently, there can be at most two transcendental singularities over $\mathbf{C}$.

When X is identified with one of the standard domains, $\mathbf{C}$ or $\mathbf{U}$, one can always ensure that the symmetry axis is $\mathbf{R}$ or $(0, 1)$. So holomorphic functions corresponding to MacLane surfaces are real functions whose all singular values are real, in other words, they are cellular functions with $\gamma = \mathbf{R}$.

It is easy to understand the nature of the cell decompositions that correspond to MacLane functions.

MacLane surfaces can be completely described by their singular values. Suppose for simplicity that the sequence of critical points on the axis is unbounded in both directions, and that all critical points are simple. Then the preimage of the real axis under p consists of the axis of symmetry and infinitely many disjoint simple curves crossing the axis at the critical points and tending to infinity in both directions. These curves together with the axis form the 1-skeleton of the cell decomposition of the plane. The sequence $(c_k)_{k \in \mathbf{Z}}$ of critical values, where each critical value is repeated according to its multiplicity satisfies

$$(c_{k+1} - c_k)(c_k - c_{k-1}) \geq 0,$$

and every sequence of critical values with this property can occur.

The corresponding class of functions is related to entire functions of Laguerre–Pólya (LP) class: these are the real entire functions which are limits of real polynomials with real zeros. According to Laguerre and Pólya, this class LP has the parametric representation:

$$z^m \exp(-az^2 + bz + c) \prod_k \left(1 - \frac{z}{a_k}\right) e^{z/a_k}, \quad a \geq 0,\ b, c, a_k \in \mathbf{R}. \tag{0.3}$$

It is easy to see that that LP-functions constitute a subclass of MacLane's class. In the case that the sequence critical values is infinite in both directions, the critical values of an LP-function satisfy one additional restriction: their signs alternate,

$$c_{k+1}c_k \leq 0. \tag{0.4}$$

5.1 MacLane's Theorem. *Every surface of MacLane's class is parabolic. The derivatives of the corresponding entire functions constitute the class LP.*

This was proved in [**37**]. A very illuminating geometric proof is given in [**51**]. We sketch Vinberg's proof, restricting ourselves for simplicity to the case when the sequence of zeros is infinite in both directions. First consider MacLane surfaces that satisfy the additional condition (0.4).

Let Ω be the domain obtained from the plane by deleting the vertical slits $L_k = \{k\pi + i(y_k - t) : 0 \leq t < \infty\}$, $k \in \mathbf{Z}$, where y_k are real numbers or $y_k = -\infty$ in which case the slit is empty. We call such domains Ω *comb domains.* We also require that the set $\{k : y_k > -\infty\}$ be unbounded in both directions. Let $\theta : H \to \Omega$ be a conformal map from the upper half-plane onto Ω, $\theta(\infty) = \infty$. Then the function $f(z) = \exp(-i\theta(z))$, initially defined in the upper half-plane, can be extended by symmetry to an entire function. It is easy to see that it belongs to MacLane's class, with critical values $c_k = (-1)^k e^{y_k}$. By approximating the region Ω by its intersections with vertical strips $\{z : |\Re z| < \pi m\}$, we obtain a sequence of real polynomials with real zeros that converge to f. Thus f is in LP class.

To prove that all functions of LP class satisfying (0.4) arise this way, we just notice that every real polynomial of degree $2m$ with all zeros real has a "comb representation" $p(z) = \exp(-i\theta(z))$ where θ is a conformal map of the upper half-plane onto a comb domain in the strip $\{z : |\Re z| < \pi m\}$. Passing to the limit we obtain a comb representation of a given LP function satisfying (0.4).

Thus we obtain a correspondence between functions of the class LP and a subclass of MacLane surfaces characterized by condition (0.4). This correspondence becomes bijective if we factor sequences of critical values by shifts, and entire functions by the change of the independent variable $az + b$, $a > 0, b \in \mathbf{R}$.

One can even tell explicitly in terms of c_k when $a = 0$ in (0.3), see [**26**], [**20**].

To prove the general case of MacLane's theorem, one can do an approximation argument similar to that in the proof of Theorem 4.2. The crucial fact here is the following.

Let (p_n) be a sequence of real polynomials whose all zeros are real, and assume that (p_n) converges in a neighborhood of 0 *to a non-zero function. Then the sequence converges on every compact subset of the plane.*

Let (X, p) be a surface of MacLane class, and $f = p \circ \phi$ the corresponding function (which is à priori holomorphic either in $\mathbf{C}$ or in $\mathbf{U}$). Using the cell decomposition, one can find an exhaustion of X by open topological discs $X_1 \subset X_2 \subset \ldots \to X$ such that each X_k is isometric to a subset of a surface corresponding to a real polynomial f_k with all critical points real. By Caratheodory's theorem, these polynomials converge to f in a neighborhood of 0. Then the derivatives f_k' are polynomials with all zeros real, and they converge in a neighborhood of 0 to f'. Applying the above proposition we conclude that f_k' converge in the whole plane, and thus f_k also converge in the whole plane. So f is entire.

The class LP and its geometric characterization occur in many questions of analysis.

The most striking application of the geometric characterization of the class LP is to the spectral theory of self-adjoint second order differential operators on the real line with periodic potentials (Hill operators). It was discovered by Krein [**35**] that Lyapunov functions of periodic strings are exactly those real entire functions f of genus zero, with positive roots, which have the property that all solutions z of the equation $f^2(z) = 1$ are non-negative.

These functions constitute a subclass of LP which can be explicitly described in terms of their comb domains.

We include a brief explanation of this connection. Consider a second-order operator

$$L(y) = -y'' + v(x)y,$$

where v is a periodic function of period π. Then the spectra of the boundary value problems with periodic or antiperiodic boundary conditions are known to be real and bounded from below.

Let $c(x, \lambda)$ and $s(x, \lambda)$ be solutions of the differential equation $L(y) = \lambda y$ with the following initial conditions:

$$s(0, \lambda) = c'(0, \lambda) = 0 \quad s'(0, \lambda) = c(0, \lambda) = 1.$$

One can show (by reducing the differential equation to an integral equation which can be solved by the standard iteration procedure) that c and s are entire functions of λ of order at most $1/2$, normal type. Following Lyapunov, we are trying to find out when all solutions of our differential equation are bounded on the real line. For this purpose, we consider the monodromy matrix

$$T(\lambda) = \begin{pmatrix} c(\pi, \lambda) & s(\pi, \lambda) \\ c'(\pi, \lambda) & s'(\pi, \lambda) \end{pmatrix},$$

whose determinant is 1 because of the Liouville's theorem and the initial conditions. One half of the trace of T is called the *Lyapunov function*,

$$u(\lambda) = (c(\pi, \lambda) + s'(\pi, \lambda))/2.$$

This is a real entire function of order at most $1/2$. The eigenvalues z_1 and z_2 of the monodromy matrix are the solutions of the quadratic equation $z^2 - 2u(\lambda)z + 1 = 0$. It is easy to see that all solutions of the differential equation will be bounded if $u^2(\lambda) \leq 1$. Thus the real line is divided into two sets: the stability zone where $u^2 \leq 1$ and the instability zone. On the boundary of the stability zone, we have $u(\lambda) = \pm 1$. Let us consider this last equation in the complex plane. If $u(\lambda) = \pm 1$, then $T(\lambda)$ has one multiple eigenvalue ± 1. This means that our differential equation $L(y) = \lambda y$ has either periodic solution of period π or an "antiperiodic" solution $y(0) = -y(\pi)$. As both periodic and antiperiodic boundary value problems are self-adjoint, all their eigenvalues should be real, and we conclude that the equation $u^2(\lambda) = 1$ has only real solutions.

So both $u - 1$ and $u + 1$ belong to the Laguerre–Pólya class (as real entire functions of order less than two, with real zeros). So all zeros of u' are also real, so u belongs to the MacLane class. Now it is easy to conclude that in fact $u \in LP$.

For Lyapunov functions, Marchenko and Ostrovski derived a different version of comb representation, namely $u(z) = \cos\theta(z)$, where θ is a conformal map of the upper half-plane onto a region obtained from the upper half-plane by deleting vertical slits from the points πk of height h_k. Thus we have

5.2 Theorem of Marchenko and Ostrovskii *Let f be a real entire function such that the equation $f^2(z) = 1$ has only real solutions. Then $f = \cos\theta$, where θ is the conformal map we just described.*

The theorem does not contain any a priori growth assumptions, so one has to show first that f is of order at most 2. This can be done by elementary Nevanlinna theory, but for applications to Lyapunov functions such generalization is unnecessary.

Krein proved the converse theorem, that every real entire function of genus zero such that all solutions of $f^2(z) - 1 = 0$ are real, is a Lyapunov function of a periodic string. The "string" in this result can be quite singular, the mass of the string is in general not a function but an arbitrary measure. So the question of characterizing Lyapunov functions arises for various subclasses when one impose conditions on the potential v, for example $v \in L^2$.

Developing Krein's ideas, Marchenko and Ostrovskii [**38**] obtained a parametrization of self-adjoint periodic Hill operators with L^2 potentials in terms of their spectral data. In a recent paper [**50**], Tkachenko extended this result to some non self-adjoint Hill operators (with complex potentials) by considering small perturbations of MacLane surfaces which are no longer symmetric, and establishing an exact correspondence between a class of entire functions and a class of surfaces in the spirit of Theorem 5.1.

We finish this section by mentioning a classical problem about real entire functions with real zeros which was recently solved.

It is evident from the definition that all derivatives of a function of the class LP have only real zeros. The converse is also true, and in a very strong sense: if f is a real entire function, and all zeros of ff'' are real, then $f \in \mathrm{LP}$ [**6**]. This was conjectured by Wiman in 1911, and the final result completes a long line of development with important contributions of Levin, Ostrovskii, Hellerstein, Williamson and Sheil-Small. Recently Edwards and Hellerstein (for the case of finite order) and Langley (for the case of infinite order) extended the result by replacing f'' by $f^{(k)}$ with any $k \geq 2$.

6. Here we mention a partial generalization of Theorem 5.1 to meromorphic functions. Consider the class of meromorphic functions of the form

$$f(z) = e^{\sigma z} \frac{\prod(1 + z/a_k)}{\prod(1 - z/b_k)}. \tag{0.5}$$

where $\sigma \geq 0$ and (a_k) and (b_k) are two increasing sequences of positive numbers, finite or infinite (possibly empty), and such that

$$\sum \frac{1}{a_k} + \sum \frac{1}{b_k} < \infty.$$

Let us denote by (x_k) the sequence of real critical points of f. They are enumerated by positive integers on the positive ray and by negative integers on the negative ray. We assume that this sequence is increasing, each point is repeated according to its multiplicity, and $x_{-1} < 0 < x_1$. Let $c_k = f(x_k) \in \overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ be the corresponding sequence of critical values.

6.1 Theorem. *All critical points of a function f of the form* (0.5) *are real. For a sequence (c_k) to be a sequence of critical values of the function of the form* (0.5), *it*

is necessary and sufficient that the following conditions are satisfied:
$(-1)^k c_k \geq 0$,
if $c_k = 0$ *then* $0 \in \{c_{k-1}, c_{k+1}\}$*; if* $c_k = \infty$ *then* $\infty \in \{c_{k-1}, c_{k+1}$*, and*
$|c_{-k}| < |c_k|$ *for all* $k > 0$ *for which both* c_k *and* c_{-k} *are defined.*

The class of meromorphic functions of the form (0.5) has another interesting parametrization: Taylor coefficients of these functions are exactly the totally positive sequences [**3**].

The class of functions (0.5) does not exhaust all real meromorphic functions whose critical points are real, even if we restrict ourselves to functions without asymptotic values.

Thus no satisfactory analog of Theorem 5.1 is known for meromorphic functions and open surfaces spread over the sphere. However, there is a related result for *rational functions* which was used in [**21**] to prove a special case of an intriguing conjecture in real algebraic geometry.

6.2 Theorem. *If all critical points of a rational function* f *belong to a circle* C *on the Riemann sphere, then* $f(C)$ *is a subset of a circle.*

Let us call two rational functions f and g equivalent if $f = \ell \circ g$ where ℓ is a fractional linear transformation. Equivalent functions have the same critical points. We may assume without loss of generality that the circle C in Theorem 6.2 is the real line. Then Theorem 6.2 says that whenever the critical points of a rational function are real, it is equivalent to a real rational function. A rational function with prescribed critical points can be obtained as a solution of a system of algebraic equations, so Theorem 6.2 implies that all solutions of this system are real whenever the coefficients are real. We will see in a moment the geometric significance of this system of algebraic equations.

It is very easy to prove Theorem 6.2 for polynomials, because every two polynomials with the same critical points are equivalent (which means that solution of the system of algebraic equations mentioned above is essentially unique in this case). This is not so for rational functions: it turns out that for given $2d-2$ points in general position on the sphere there are finitely many, namely

$$u_d = \frac{1}{d}\binom{2d-2}{d-1}, \quad \text{the } d\text{-th Catalan number,}$$

of classes of rational functions of degree d which share these critical points. This is due to L. Goldberg [**28**] who reduced the problem to the following classical problem of enumerative geometry: given $2d-2$ lines in general position in (complex) projective space, how many subspaces of codimension 2 intersect all these lines? The answer to this last problem, the Catalan number u_d, was obtained by Schubert in 1886 who invented what is now known as "Schubert Calculus" to solve this and similar enumerative problems.

Notice that rational functions exhibit a very special property in Goldberg's result: as we mentioned above, there is only one class of polynomials with prescribed critical points, and similarly there is only one class of Blaschke products with prescribed critical points [**54**].

The general question of how many solutions to a problem of enumerative geometry can be real was asked by Fulton in [**24**]. A specific conjecture about the problem of finding subspaces of appropriate codimension intersecting given real subspaces was made by B. and M. Shapiro. For the problem of Schubert calculus stated above, this conjecture says that if all $2d-2$ given lines are tangent to the rational normal curve $z \mapsto (1 : z : \ldots : z^d)$ at real points, then all u_d subspaces of codimension 2 which intersect these lines are real (can be defined by real equations). This statement is equivalent to Theorem 6.2.

The proof of Theorem 6.2 is based on an explicit description of surfaces spread over the sphere which correspond to real rational functions with real critical points in the spirit of Vinberg's work about the MacLane's class mentioned in section 4.

Let R be the class of *real* rational functions f whose all critical points are real and simple. To describe the Riemann surface of f^{-1} we consider the *net* γ_f which is the preimage $\gamma_f = f^{-1}(\mathbf{R} \cup \infty)$ modulo homeomorphisms of the Riemann sphere commuting with complex conjugation. A net consists of simple arcs which meet only at the critical points of f. To each of these arcs we prescribe a *label* equal to the length of its image under f. One can describe explicitly all labeled nets which may occur from this construction. It turns out that labeled nets give a parametrization of the class R of rational functions. This parametrization has an advantage that it clearly separates the discrete, topological parameter (the net) from the continuous parameters (the labeling), and it turns out that the set of possible labelings of a given net has simple topological structure: it is a convex polytope.

Using topological methods, we show in [**21**] that for every net and for every set of $2d-2$ points on the real line there exists a rational function of the class R with this net and these critical points. On the other hand, a simple combinatorial argument shows that the number of possible nets on $2d-2$ vertices is equal to the Catalan number u_d. Thus one obtains u_d classes of real rational functions of degree d with prescribed real critical points. Comparison with L. Goldberg's result shows that in fact we constructed *all* classes of rational functions with prescribed real critical points. Thus all such classes contain real functions.

Since the publication of [**21**], two new proof of Theorem 6.1 appeared. One is a very much simplified version of the original proof; it still uses the parametrization of the class R described above, but avoids the Uniformization theorem [**22**]. Another proof [**40**] is based on completely different ideas, coming from the theory of exactly solvable models of statistical mechanics. It proves not only Theorem 6.1 but also its multi-dimensional generalization to rational curves in projective spaces.

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References

[1] L. Ahlfors, *Sur les fonctions inverses des fonctions méromorphes,* Comptes Rendus, **194** (1932) 1145–1147. Collected papers, v. 1, Birkhäuser, Boston, 1982, 149–151.

[2] L. Ahlfors, *Zur Theorie der Überlagerungsflächen,* Acta Math., **65** (1935) 157–194. Collected papers, v. 1, Birkhäuser, Boston 1982, 214–251.

[3] M. Aissen, I. Schoenberg, A. Whitney, On the generating functions of totally positive sequences. I. J. Analyse Math. 2, (1952). 93–103.

[4] W. Bergweiler, *A new proof of the Ahlfors five islands theorem,* J. Analyse Math., **76** (1998) 337–347.

[5] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order,* Revista mat. Iberoamer., **11** (1995) 355–373.

[6] W. Bergweiler, A. Eremenko and J. Langley, *Real entire functions of infinite order and a conjecture of Wiman,* GAFA, 13 (2003), 975–991.

[7] W. Bergweiler and A. Eremenko, Direct singularities and completely invariant domains of entire functions, Illinois J. Math.

[8] A. Bloch, *Les théorèmes de M. Valiron sur les fonctions entières, et la théorie de l'uniformisation,* Comptes Rendus, **178** (1924) 2051–2052.

[9] A. Bloch, *La conception actuelle de la théorie des fonctions entières et méromorphes,* Enseign. Math., **25** (1926) 83–103.

[10] M. Bonk, *Singular surfaces and meromorphic functions,* Notices AMS, **49** 6 (2002) 647–657.

[11] M. Bonk and A. Eremenko, *Covering properties of meromorphic functions, negative curvature and spherical geometry,* Ann. Math., **152** (2000) 551–592.

[12] M. Bonk and A. Eremenko, *Uniformly hyperbolic surfaces,* Indiana Univ. Math. J., **49** (2000) 61–80.

[13] D. Drasin and A. Weitsman, *Meromorphic functions with large sums of deficiencies,* Adv. Math., **15** (1975) 93–126.

[14] P. Doyle, *Random walk on the Speiser graph of a Riemann surface,* Bull. AMS, **11** (1984) 371–377.

[15] J. Dufresnoy, *Sur les domaines couvertes par les valeurs d'une fonction méromorphe ou algébroïde,* Ann. Sci. École Norm. Super., **58** (1941) 179–259.

[16] A. Edrei, On the generating functions of totally positive sequences. II. J. Analyse Math. 2, (1952). 104–109.

[17] G. Elfving, *Über eine Klasse von Riemannschen Flächen und ihre Uniformisierung,* Acta Soc. Sci. Fenn., **2** (1934) 1–60.

[18] A. Eremenko, *On the set of asymptotic values of a meromorphic function of finite order,* Engl. transl.: Mat. Notes, **24** (1979), 914–916.

[19] A. Eremenko, On the entire functions bounded on the real axis, Soviet Math. Dokl., 37, 3 (1988), 693-695.

[20] A. Eremenko, Critical values of generating functions of totally positive sequences, Journal of Math. Physics, Analysis and Geometry, 11 (2004) 1-13.

[21] A. Eremenko and A. Gabrielov, *Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry,* Ann. Math., **155** (2002) 105–129.

[22] A. Eremenko and A. Gabrielov, Elementary proof of the B. and M. Shapiro conjecture for rational functions, arXiv:math/0512370.

[23] A. Eremenko and P. Yuditskii, Uniform approximation of sgn(x) by polynomials and entire functions. J. d'Analyse Math., 101 (2007) 313-324.

[24] W. Fulton, *Introduction to intersection theory in algebraic geometry*, AMS, Providence, RI, second edition, 1996.

[25] A.A. Gol'dberg, *On a class of Riemann surfaces,* Mat. Sbornik, **49** (1959) 447–458.

[26] A.A. Gol'dberg, *Some asymptotic properties of meromorphic functions,* Complex Variables, **37** (1998) 225–241. (Translated from the Russian original of 1956).

[27] A.A. Gol'dberg and I.V. Ostrovskii, *Distribution of values of meromorphic functions*, Moskva, Nauka, 1970 (Russian, Eglish transl. to appear in AMS).

[28] L. Goldberg, *Catalan numbers and branched coverings of the sphere,* Adv. Math., **85** (1991) 129–144.

[29] W. Gross, *Über die Singularitäten analytischer Funktionen,* Monatshefte Math. Phys., **29** (1918).

[30] M. Heins, *The set of asymptotic values of an entire function,* 12-e Skand. Matematikerkongr., Lund 1953, Lund, 1954, 56–60.

[31] M. Heins, *Asymptotic spots of entire and meromorphic functions,* Ann. Math., **66** (1957) 430–439.

[32] S. Hellerstein and J. Williamson, *Derivatives of entire functions and a question of Pólya, I,* Trans. AMS, **227** (1977) 227–249; II, ibid., 234 (1977) 497–503.

[33] F. Iversen, *Recherches sur les fonctions inverses des fonctions méromorphes*, Thèse, Helsingfors, 1914.

[34] B.Ya. Levin and I.V. Ostrovskii, *The dependence of the growth of an entire function on the distribution of zeros of its derivative,* Sibirsk. Mat. Z., 1 (1960) 427–455. English transl., AMS Transl. (2) 23 (1963), 323-357.

[35] M.G. Krein, *On the inverse problems of the theory of filters and λ-zones of stability,* Doklady AN SSSR, **93** (1953) 767–770 (Russian).

[36] H. Künzi, *Zwei Beispiele zur Wertverteilungslehre,* Math. Z., **62** (1955) 94–98.

[37] G. MacLane, *Concerning the uniformization of certain Riemann surfaces allied to the inverse-cosine and inverse-gamma surfaces,* Trans. AMS, **62** (1947) 99–113.
[38] V.A. Marchenko and I.V. Ostrovskii, *A characterization of the spectrum of the Hill operator,* Math. USSR Sbornik **26** 4 (1975) 493–554. (English transl.)
[39] S. Mazurkiewicz, *Sur les points singuliers d'une fonction analytique,* Fund. Math., (1931) 26–29.
[40] Mukhin, Tarasov and Varchenko, The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz, arXiv:math/0512299.
[41] R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer-Verlag, Berlin, 1953.
[42] R. Nevanlinna, *Über Riemannsche Flächen mit endlich vielen Windungspunkten,* Acta Math., **58** (1932) 295–373.
[43] Ch. Pommerenke, *Boundary behavior of conformal maps*, Springer, Berlin, 1992.
[44] Ch. Pommerenke, *Estimates for normal meromorphic functions,* Ann. Acad. Sci. Fenn., Ser A **476** (1970).
[45] Yu.G. Reshetniak, *Two-dimensional manifolds of bounded curvature*, Geometry IV, Encyclopaedia of Math. Sci., 70, Springer, Berlin, 1993.
[46] R. Robinson, A generalization of Picard's and related theorems, Duke Math. J., 5 (1939) 118-132.
[47] T. Sheil-Small, *On the zeros of the derivatives of real entire functions and Wiman's conjecture,* Ann. Math., **129** (1989) 170–193.
[48] Y. Sibuya, Global theory of a second order linear ordinary differential equation with a polynomial coefficient, North Holland, Amsterdam, 1975.
[49] O. Teichmüller, *Eine Anwendung quasikonformer Abbildungen auf das Typenproblem,* Deutsche Math., **2** (1937) 321–327. Gesammelte Abhandlungen, Springer, Berlin, 1982, 171–177.
[50] V. Tkachenko, *Spectra of non-selfadjoint Hill's operators and a class of Riemann surfaces,* Ann. Math., **143** (1996) 181–231.
[51] E.B. Vinberg, *Real entire functions with prescribed critical values,* Problems of group theory and homological algebra, Yarosl. Gos. Un-t., Yaroslavl, 1989, 127–138 (Russian).
[52] L.I. Volkovyski, *Researches on the type problem of a simply connected Riemann surface*, Trudy Mat. Inst. im. V.A. Steklova, **34** 1950 (Russian).
[53] H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Springer-Verlag, Berlin, 1955.
[54] S. Zakeri, *On critical points of proper holomorphic maps of the unit disc,* Bull. LMS, **30** (1998) 62–66.

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