MATH 303, Fall 2018 Midterm exam

NAME: A. Eremenko

1. For the differential equation

$$(x^5 - 2x^4 + x^3)y'' + 2x^2y' + (x-1)y = 0,$$

find all singular points and determine which of them are regular.

Solution. Singular points are 0 and 1, both irregular.

2. a) The differential equation

$$(x^3+1)y''+xy'+y=0$$

has a solution

$$y(z)=1+\sum_{n=1}^{\infty}a_n(x-2)^n. \tag{1}$$

Give a lower estimate of the radius of convergence of this series, based on the location of singularities,

b) The differential equation

$$(x-1)^2y''+x(x-1)y'+y=0$$

has a solution of the same form (1). Determine *precisely* the radius of convergence of the series. (Give a *full justification* of your answer.)

Solution a). To find singular points one has to solve $x^3+1=0$. One root $x_1=-1$ is evident. To find two other roots, we factor

$$x^3+1=(x+1)(x^2-x+1).$$

Solving the quedratic equation we obtain the roots

$$x_{2,3}=1/2\pm i\sqrt{3}/2.$$

The radius of convergence is **at least** the distance from the point 2 to the closest singular point. The distance from 2 to $1/2+i\sqrt{3}/2$ equals

$$\sqrt{(2-1/2)^2+3/4}=\sqrt{9/4+3/4}=\sqrt{12}/2=\sqrt{3}.$$

The distance to the other two roots is greater than or equal to this. Therefore the lower estimate for the radius of convergence is $\sqrt{3}$.

b) There is only one singular point, $x_1=1$, and the distance from 2 to 1 is 1. This gives the **lower** estimate for the radius of convergence. To see whether it is exact, write and solve the indicial equation at the point 1. The equation is

$$r(r-1)+r+1=0,$$

and its solutions are $\pm i$. From the general theory we conclude that the general solution of our equation has the form

$$c_1 \cos \log |x-1| + c_2 \sin \log |x-1|.$$

This function cannot have limit as $x \to 1$, unless $c_1 = c_2 = 0$. But $c_1 = c_2 = 0$ is incompatible with the initial condition $y(0) = 1$. Thus we know that opur solution cannot have a limit as $x \to 1$, therefore the radius of convergence is exactly 1.

3. The differential equation

$$y'' + (2 + x)y' - 2y = 0$$

has a solution of the form

$$y(x) = 1 + \sum_{n=2}^{\infty} a_n x^n.$$

Find a_2, a_3 and a_4.

Solution. First method. We know from the given form of the solution that $y(0) = 1,\ y'(0) = 0$. Plugging $x = 0, y(0) = 1, y'(0) = 0$ to the equation, we obtain that $y''(0) = 2$.

Differentiating the equation we obtain

$$y''' = -(2 + x)y'' + y'.$$

Plugging to this $x = 0, y(0) = 1, y'(0) = 0, y''(0) = 2$, we obtain $y'''(0) = -4$.

Differentiating again we obtain

$$y^{IV} = -(2 + x)y'''.$$

Plugging $x = 0, y'''(0) = 4$, we obtain $y^{IV}(0) = 8$.

Now Taylor's formula gives $a_2 = y''(0)/2! = 1,\ a_3 = y'''(0)/3! = -2/3$, and $a_4 = y^{IV}(0)/4! = 1/3$.

Second method. Write

$$y(x) = 1 + a_2x^2 + a_3x^3 + a_4x^4 + \ldots,$$

and differentiate twice:

$$y'(x) = 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots,$$

$$y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \ldots.$$

Plugging this to the equation and balancing the terms with powers $0, 1, 2$ of x, we obtain $a_2 = 1,\ a_3 = -2/3,\ a_3 = 1/3$.

4. Find the number c such that the solution of the initial value problem

$$x^2y'' - 2y = 0, \quad y(1) = 1, \quad y'(1) = c$$

remains bounded as $x \to 0$.

Solution. This is an Euler equation. The indicial equation is $r(r-1)-2 = r^2 - r - 2 = 0$. Solving it we obtain $r_1 = 2,\ r_1 = -1$. So the general solution of our differential equation is

$$y(x) = c_1x^2 + c_2x^{-1}.$$

Plugging the initial condition, we obtain a system

$$c_1 + c_2 = 1, \quad 2c_1 - c_2 = c.$$

Solving this system we obtain $c_1 = (c+1)/3, \quad c_2 = (2-c)/3$.

Solution y will be bounded as $x \to 0$ if and only if $c_2 = 0$. Because $x^{-1} \to \infty$ when $x \to 0$ and the other summand is bounded. Therefore, the necessary and sufficient condition for boundedness of our solution is $c = 2$.

5. Solve using the Laplace transform:

$$y'' + 3y' + 2y = f(t), \quad y(0) = y'(0) = 0,$$

where f is defined by

$$f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & 1 \leq t < \infty, \end{cases}$$

Solution. Laplace transform of the right hand side is

$$\int_0^1 e^{-st} dt = (1 - e^{-s})/s.$$

(The function in the RHS is $1 - u_1$ so you could also use the table). Taking the Laplace transform of the equation we obtain

$$Y(s) = \frac{1 - e^{-s}}{s(s^2 + 3s + 2)} = \frac{1 - e^{-s}}{s(s+1)(s+2)}.$$

Now use the partial fraction decomposition:

$$\frac{1}{s(s^2 + 3s + 2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}.$$

Using the table backwards we find

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u_1(t)\left(\frac{1}{2} - e^{1-t} - \frac{1}{2}e^{-2(t-1)}\right).$$

6. Find the inverse Laplace transforms of the function

$$F(s) = \frac{1}{(2s-1)^2} + \frac{e^{-2s}}{2s^2+1}.$$

Solution. From the tables we find

$$f(t) = \frac{te^{t/2}}{4} + \frac{1}{\sqrt{2}} \sin \frac{t-2}{\sqrt{2}}.$$