Elementary proof of the B. and M. Shapiro conjecture for rational functions

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Abstract

We give a new elementary proof of the following theorem: if all critical points of a rational function g belong to the real line then there exists a fractional linear transformation ϕ such that $\phi \circ g$ is a real rational function.

One of the many equivalent formulations of the Shapiro conjecture recently proved by Mukhin, Tarasov and Varchenko is the following. Let $\mathbf{f} = (f_1, \ldots, f_p)$ be a vector of polynomials in one complex variable, and assume that the Wronski determinant $W(\mathbf{f}) = W(f_1, \ldots, f_p)$ has only real roots. Then there exists a matrix $A \in \mathrm{GL}(p, \mathbf{C})$ such that $\mathbf{f}A$ is a vector of real polynomials.

This result plays an important role in real enumerative geometry [16, 17], theory of real algebraic curves [8] and has applications to control theory [12, 4]. We refer to a survey [18] and the book [19] for a comprehensive discussion of this result and related problems.

In [1] we proved the Shapiro conjecture in the first non-trivial case $p = 2$. The main idea was a coding of the elements of the preimage of the Wronski map with certain combinatorial objects. The proof in [1] was quite complicated, and its main drawback was the non-constructive character, and especially the use of the Uniformization theorem. Later we found a simpler proof [5] for $p = 2$ which preserves the main idea of [1] but employs only elementary arguments.

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The first proof of the general statement in arbitrary dimension is [9]. Soon after that a new proof appeared [10]. These proofs of the general Shapiro conjecture explore the connection with integrable models of mathematical physics.

This paper is a modified version of our preprint [5]. The reasons why we publish this special case is that on our opinion[1] this proof has one substantial advantage in comparison with all known proofs of the general case: it provides a canonical labeling of the preimage of the Wronski map by certain combinatorial objects which we call the nets. This coding was used in [6], for example, to prove the so-called "secant conjecture" whose higher dimensional analog remains unknown. A canonical coding of the preimage of the Wronski map can be also useful in the study of monodromy of this map, see [11].

Consider a non-constant rational function $g = f_1/f_2$, and assume that the polynomials f_1 and f_2 are co-prime. Then the degree of g is given by $d = \max\{\deg f_1, \deg f_2\}$, and the roots of the Wronski determinant $W(\mathbf{f}) = f_1 f_2' - f_1' f_2$ coincide with the critical points of g. Let us call two rational functions g_1 and g_2 equivalent if $g_1 = \phi \circ g_2$ for some fractional-linear transformation ϕ. Evidently, equivalent rational functions have the same critical points. So our result is

Theorem 1. *If all critical points of a rational function are real then it is equivalent to a real rational function.*

It is enough to prove this theorem for rational functions with simple critical points. The general case then follows by a limiting process.

It is known [7] that for given $2d-2$ points in the complex plane in general position, there exist

$$u_d = \frac{1}{d} \binom{2d-2}{d-1}, \tag{1}$$

the d-th Catalan number of classes of rational functions of degree d with these critical points. It turns out that the general position assumption in this result can be removed if the given points are real. Moreover, the following result turns out to be equivalent to Theorem 1:

Theorem 2. *For any given $2d-2$ distinct points on the real line, there exist exactly u_d distinct classes of rational functions of degree d with these critical*

[1]shared by F. Sottile, [18, p. 54]

points.

It follows from Theorem 1 that each of these u_d classes contains a real function. The assumption that the critical points are real is essential in Theorem 2: for $2d-2$ complex points, the number of rational functions of degree d with these critical points can be less than u_d.

Equivalence of theorems 1 and 2 was known for some time, see, for example, [17].

To state a generalization of Theorem 2 to the case of multiple critical points, we recall the definition of *Kostka numbers.* Let $\mathbf{a} = (a_1, \ldots, a_q)$ be a vector of integers satisfying

$$1 \leq a_j \leq d-1, \quad \sum_{j=1}^{q} a_j = 2d-2. \tag{2}$$

Consider the Young diagrams of shape $2 \times (d-1)$. They consist of two rows of length $d-1$. A *semi-standard Young tableau* SSYT of shape $2 \times (d-1)$ is a filling of such a diagram by positive integers, such that an integer k appears a_k times, the entries are strictly increasing in columns and non-decreasing in rows. The corresponding Kostka number $K_{\mathbf{a}}$ is the number of such SSYT. The number $K_{\mathbf{a}}$ does not change if the coordinates of $\mathbf{a}$ are permuted [20, Thm. 7.10.2].

Theorem 3. *For given* $\mathbf{a}$ *satisfying* (2)*, and given real points* $x_1 < x_2 < \ldots < x_q$*, there are exactly* $K_{\mathbf{a}}$ *classes of rational functions of degree* d *with critical points at* x_j *of multiplicity* a_j*.*

We obtain Theorem 2 as a special case when $q = 2d-2$ and all $a_j = 1$. Theorem 3 is true for *generic* complex x_j; this was derived by Scherbak [13] from the results in [21]. Theorem 3 was first proved in [6], where a result from [1] was used. We include a proof here to show that it can be achieved with the same elementary tools as theorems 1 and 2, and no heavy machinery from [1] is needed.

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1 Wronski map

We recall the necessary facts and definitions. Let $\mathbf{f} = (f_1, f_2)$ be a pair of linearly independent polynomials of degree at most d. They span a 2-dimensional subspace in the space of all polynomials of degree at most d, and thus define a point in the Grassmannian $G = G(2, d+1)$. Two pairs of polynomials are equivalent if they span the same subspace.

The Wronski determinants of equivalent pairs are proportional non-zero polynomials of degree at most $2d-2$. Classes of proportionality of such polynomials form a space Poly^{2d-2} which can be identified with the projective space $\mathbf{P}^{2d-2}$. So taking Wronski determinant defines a map

$$W : G(2, d+1) \to \mathrm{Poly}^{2d-2} \tag{3}$$

which is called the *Wronski map*. The real Grassmannian $G_{\mathbf{R}}$ or the real projective space $\mathrm{Poly}_{\mathbf{R}}$ consist of those points whose coordinates (coefficients of the polynomials) can be chosen real. It is clear that W sends $G_{\mathbf{R}}$ to $\mathrm{Poly}_{\mathbf{R}}^{2d-2}$. The Wronski map is a finite regular map of compact algebraic manifolds, and its degree can be defined as the number of preimages of a generic point. This number turns out to be the Catalan number u_d, see, for example, [7]. If the Grassmannian G is embedded in a projective space by the Plücker embedding, then the Wronski map W becomes a restriction of a linear projection on G. Thus the degree of W is the same as the degree of the Grassmann variety, that is the number of intersections of a generic subspace of codimension $2d-2$ with the Plücker embedding of G.

Using this notation, theorems 1 and 2 can be restated as follows:

1. *The full preimage of a polynomial with all real roots under the Wronski map consists of real points in G.*

2. *Every polynomial in* Poly^{2d-2} *with distinct real roots has exactly u_d distinct preimages under the Wronski map.*

If a pair (f_1, f_2) represents a point of G, then $g = f_1/f_2$ is a non-constant rational function of degree at most d. If two pairs of polynomials represent the same point of G, then the corresponding rational functions are equivalent. This defines a map r from G into the set Rat^d of equivalence classes of non-constant rational functions of degree at most d. This map is not injective because polynomials in a pair can have a common factor. More precisely, let $Z_0 \subset G$ be the locus of points corresponding to pairs of polynomials having a non-constant common factor, and Z_1 the locus corresponding to polynomials of degree less than d. Put $Z = Z_0 \cup Z_1$. Then

$$r : G\backslash Z_0 \to \mathrm{Rat}^d \tag{4}$$

is a bijection. The standard topology on G can be defined as induced by the Plücker embedding, and the topology on Rat^d is of uniform convergence with respect to the spherical metric. The map r is continuous on $G\backslash Z$ but not continuous on the whole G. The following weaker continuity property of r holds on the whole Grassmannian.

Proposition 1. *Let (p_j) be a converging sequence in G, $p_j \in G\backslash Z$ and $p = \lim p_j$ is represented by a pair of polynomials with a common factor q. Let $z_1, \ldots, z_k$ be the roots of q. Then $r(p_j) \to r(p)$ uniformly on compact subsets of $\mathbf{C}\backslash\{z_1, \ldots, z_k, \infty\}$. One does not have to include ∞ if the degree of p is the same as that of p_j.*

The elementary proof is left to the reader.

2 Nets of rational functions

Let R^d be the class of real non-constant rational functions g of degree at most d whose all critical points are real. Consider the full preimage $\gamma = g^{-1}(\overline{\mathbf{R}})$, where $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. This preimage consists of simple analytic arcs which meet only at critical points. These arcs define a cell decomposition $C(g)$ of the Riemann sphere $\mathbf{P}^1$ whose 2-cells (faces) are components of $\mathbf{P}^1\backslash\gamma$, 1-cells (edges) are components of $\gamma\backslash\{\text{critical points}\}$ and 0-cells (vertices) are the critical points. We choose some vertex v_0 and call it the *distinguished vertex* of $C(g)$. The union γ of edges and vertices is the 1-skeleton of the cell decomposition. Such a cell decomposition $C = C(g)$ has the following properties:

(i) the 1-skeleton of C contains $\overline{\mathbf{R}}$.
(ii) C is symmetric with respect to $\overline{\mathbf{R}}$,
(iii) all vertices belong to $\overline{\mathbf{R}}$ and an even number of edges meet at each vertex.

If $C = C(g)$ the even number in (iii) is twice the local degree of g at the critical point. Another important property of our cell decomposition is that the closure of every cell is homeomorphic to a closed ball of the same dimension. It follows that

(iv) no edge can begin and end at the same point.

Two cell decompositions C_1 and C_2 satisfying (i)–(iv) and having distinguished vertices v_0^1 and v_0^2 will be called *equivalent* if there exists a homeomorphism $\phi : \mathbf{P}^1 \to \mathbf{P}^1$ commuting with reflection with respect to $\overline{\mathbf{R}}$, preserving orientations of $\overline{\mathbf{R}}$ and $\mathbf{P}^1$, mapping cells of C_1 onto cells of C_2 and v_0^1 to v_0^2. An equivalence class of cell decompositions satisfying (i)–(iv) will be called a *net*. The number of faces of a net is even, we denote it by $2d$ and call the positive integer d *the degree of the net.* If $C = C(g)$ then $\deg C = \deg g$. A net of degree d has $2d - 2$ edges disjoint from the real axis. Using the Uniformization theorem, we proved in [1, 2] that each net comes from a rational function, and the critical points of this rational function can be arbitrarily prescribed, but we do not use this result here, and in fact it will be deduced from our theorems 1 and 2 in the end of Section 1.

We need the following elementary

Proposition 2. *Let (p_j) be a convergent sequence in $G_{\mathbf{R}}$, and $p = \lim p_j$. Let $g_j = r(p_j)$ be the corresponding sequence of rational functions. Then the sets $\gamma_j = g_j^{-1}(\overline{\mathbf{R}})$ converge in the Hausdorff metric to the set $\gamma = g^{-1}(\overline{\mathbf{R}})$, where $g = r(p)$.*

This is a simple corollary of Proposition 1, and the details are left to the reader.

Corollary 1. *Suppose that $g_t : t \in [0, 1]$ is a continuous path in R^d and each g_t has $2d - 2$ simple critical points. Let one of these critical points be $v_0(t)$, a continuous function of t. Then the net of g_t with distinguished point $v_0(t)$ is independent of t.*

Corollary 2. *Let p_t be a continuous path in the Grassmannian G, parametrized by $[0, 1]$. Suppose that $g_t = r(p_t)$ belong to R^d and have critical points $x_0(t), \ldots, x_n(t)$, such that $x_j(t) \neq x_i(t)$ for $0 \leq j < i \leq n$ and $0 \leq t < 1$, while for $t = 1$ we have $x_0(1) = x_1(1)$ and $x_j(t) \neq x_i(t)$ for $1 \leq j < i \leq n$.*

Then the degrees of g_t are equal for $t \in [0, 1)$, and the degree of g_1 is less than the degree of g_t, $t \in [0, 1)$ if and only if the net of g_0 contains an edge from x_0 to x_1.

Proof. According to Proposition 2, the sets $\gamma_t = g_t^{-1}(\overline{\mathbf{R}})$ vary continuously in the Hausdorff metric. If there is an edge connecting $x_0(t)$ and $x_1(t)$, the limit of this edge as $t \to 1$ cannot be a loop because of the property (iv) of the nets. So the limit belongs to $\overline{\mathbf{R}}$ and thus the limit cell decomposition has fewer faces than the cell decomposition $C(g_1)$.

In the opposite direction, if this limit has fewer faces, some edge has to disappear in the limit, and this can only be an edge from x_0 to x_1. □

3 Thorns

Let $X^{2d-2} \subset \mathrm{Poly}_{\mathbf{R}}^{2d-2}$ be the subset consisting of polynomials whose all roots are real. Then X^{2d-2} has non-empty interior. Here we construct an open subset of X^{2d-2} such that for every polynomial p in this subset, the full preimage $W^{-1}(p)$ consists of u_d distinct real points in G. The existence of such a subset was established by Sottile [15], but we give a more precise description of this set following [3].

We fix $d \geq 2$. Consider pairs of integers

$$0 \leq k_1 < k_2 \leq d \tag{5}$$

and pairs $\mathbf{q} = (q_1, q_2)$ of real polynomials

$$\begin{aligned} q_1(z) &= \phantom{z^d + a_{2,d-1}} z^{d-1} + a_{1,d-2}z^{d-2} + \ldots + a_{1,k_1}z^{k_1}, \\ q_2(z) &= z^d + a_{2,d-1}z^{d-1} + \ldots + a_{2,k_2}z^{k_2}. \end{aligned} \tag{6}$$

Suppose that all coefficients $a_{i,j}$ are strictly positive, all roots of the Wronskian determinant $W(\mathbf{q}) = W(q_1, q_2)$ belong to the semi-open interval $(-1, 0] \subset \mathbf{R}$, and those roots on the open interval $(-1, 0)$ are simple. The set of all such polynomial pairs (6) will be denoted by $b(k_1, k_2)$. The greatest common factor of $\{q_1, q_2\}$ is z^{k_1}.

It is easy to see that the representation of a point of $G_{\mathbf{R}}$ by a pair in $b(k_1, k_2)$ is unique. Setting $k_1 = 0$ and $k_2 = 1$ we obtain an open subset $b(0, 1)$ of $G_{\mathbf{R}}$.

The multiplicity of the root of $W(\mathbf{q})$ at 0 is $k = k_1 + k_2 - 1$. We enumerate the negative roots of $W(\mathbf{q})$ as

$$-x_{2d-2} < -x_{2d-3} < \ldots < -x_{k+1} < 0. \tag{7}$$

Let ϵ be a positive increasing function on $[0,1]$ satisfying $\epsilon(0)=0,\ \epsilon(x)<x$, for $x\in(0,1]$. The set of all such functions will be denoted by E. A *thorn* T of dimension n is a region in $\mathbf{R}^n$ of the form

$$T(n,\epsilon)=\{(y_1,\ldots,y_n):0<y_n<\epsilon(1),\ 0<y_k<\epsilon(y_{k+1}),\ 1\leq k\leq n-1\}.$$

Let $w(k,T)$ be the set of polynomials of the form

$$p(x)=x^k(x+x_{2d-2})(x+x_{2d-3})\ldots(x+x_{k+1}), \tag{8}$$

where the vectors $(x_{2d-2},\ldots,x_{k+1})$ belong to a thorn T of dimension $2d-2-k$.

The set $b(d-1,d)$ consists of a single pair $q_1=z^{d-1},\ q_2=z^d$.

Consider the following two operations $F^i,\ i=1,2$. For each pair (q_1,q_2), of the form (6) operation F^i adds to the polynomial q_i one term az^{k_i-1}, where $a>0$ is a small parameter, and leaves the other polynomial of the pair unchanged. So each operation F^i increases the total number of non-zero coefficients of a polynomial pair (6) by one:

$$F^i_a:b(\mathbf{k})\to b(\mathbf{k}-\mathbf{e}_i),\quad F^i_a(\mathbf{q})=\mathbf{q}+az^{k_i-1}\mathbf{e}_i, \tag{9}$$

where $\mathbf{k}=(k_1,k_2)$ and $(\mathbf{e}_1,\mathbf{e}_2)$ is the standard basis in $\mathbf{R}^2$, and $a>0$ is a small parameter whose range may depend on $\mathbf{q}$. The following rule has to be observed:

Rule. *Operation F^i is permitted on $b(k_1,k_2)$ if and only if the outcome of this operation does not violate inequalities* (5).

In other words, operation F^1 is permitted on $b(k_1,k_2)$ if $k_1>0$, and operation F^2 is permitted on $b(k_1,k_2)$ if $k_2>k_1+1$.

The following result, which is a part of [3, Proposition 8] shows, among other things, that the F^i are well defined, that is their result indeed belongs to some $b(k_1^*,k_2^*)$ for sufficiently small values of the parameter a.

Proposition 3. *Suppose that $\mathbf{k}=(k_1,k_2)$ and $i\in\{1,2\}$ satisfy the Rule above. Suppose that for some thorn T of dimension $2d-2-k$ a set $U\subset b(k_1,k_2)$ is given, such that the map $\mathbf{q}\mapsto W(\mathbf{q}):U\to w(k,T)$ is surjective. Then there exists a thorn T^* of dimension $2d-1-k$ and a set $U^*\subset b(\mathbf{k}-\mathbf{e}_i)$, such that every $\mathbf{q}^*\in U^*$ has the form $F^i_a(\mathbf{q}),\ \mathbf{q}\in U$, where F^i_a is defined in* (9), *and $a>0$, and the map*

$$\mathbf{q}^*\mapsto W(\mathbf{q}^*):U^*\to w(k-1,T^*) \tag{10}$$

is surjective.

Proof. We follow [3, Section 2]. First we state three elementary lemmas about thorns.

Lemma 1. *Intersection of any finite set of thorns of same dimension contains a thorn of the same dimension.*

Proof. Take the minimum of their defining functions. □

Lemma 2. *Let $T = T(n, \epsilon)$ be a thorn of dimension n in $\mathbf{R}^n = \{(x_1, \ldots, x_n)\}$, and U its neighborhood in $\mathbf{R}^{n+1} = \{(x_0, \ldots, x_n)\}$. Then $U^+ = U \cap \mathbf{R}^{n+1}_{>0}$ contains a thorn $T(n+1, \epsilon_1)$.*

Proof. There exists a continuous function $\delta_0 : T \to \mathbf{R}_{>0}$, such that U^+ contains the set $\{(x_0, \mathbf{x}) : \mathbf{x} \in T, 0 < x_0 < \delta_0(\mathbf{x})\}$. Let $\delta(t)$ be the minimum of δ_0 on the compact subset $\{\mathbf{x} \in \overline{T(n, \epsilon/2)} : x_1 \geq t\}$ of T. Then there exists $\epsilon_0 \in E$ with the property $\epsilon_0 < \delta$. If we define $\epsilon_1 = \min\{\epsilon/2, \epsilon_0\}$, then $T(n+1, \epsilon_1) \subset U^+$. □

Lemma 3. *Let $T = T(n+1, \epsilon)$ be a thorn of dimension $n+1$, and $h : T \to \mathbf{R}^{n+1}_{>0}$, $(x_0, \mathbf{x}) \mapsto (y_0(x_0, \mathbf{x}), \mathbf{y}(x_0, \mathbf{x}))$, a continuous map with the properties: for every $\mathbf{x}$ such that $(x_0, \mathbf{x}) \in T$ for some $x_0 > 0$, the function $x_0 \mapsto y_0(x_0, \mathbf{x})$ is increasing, and $\lim_{x_0 \to 0} \mathbf{y}(x_0, \mathbf{x}) = \mathbf{x}$. Then the image $h(T)$ contains a thorn.*

Proof. We consider the region $D \in \mathbf{R}^{n+1}$ consisting of T, its reflection T' in the hyperplane $x_0 = 0$ and the interior with respect to this hyperplane of the common boundary of T and T'. The map h extends to T' by symmetry: $h(-x_0, \mathbf{x}) = -h(x_0, \mathbf{x}), (x_0, \mathbf{x}) \in T$, and then to the whole D by continuity. It is easy to see that the image of the extended map contains a neighborhood U of the intersection of D with the hyperplane $x_0 = 0$. This intersection is a thorn T_1 in $\mathbf{R}^n = \{(x_0, \mathbf{x}) \in \mathbf{R}^{n+1} : x_0 = 0\}$. Applying Lemma 2 to this thorn T_1, we conclude that U^+ contains a thorn. □

We continue the proof of Proposition 3.

Let us fix $\mathbf{q} \in U$, and put $W = W_{\mathbf{q}}$. As $W \in w(k, \epsilon)$, we have $\operatorname{ord} W = k$, where ord denotes the multiplicity of a root at 0. Let cz^k be the term of the smallest degree in $W(z)$. Then $c > 0$, because all roots of W are non-positive. In fact,

$$c = (k_2 - k_1) a_{2,k_2} a_{1,k_1} > 0. \tag{11}$$

We fix $i \in \{1,2\}$ satisfying the Rule above and define $W^* = W_{\mathbf{q}^*}$, where $\mathbf{q}^* = F_a^i(\mathbf{q})$. Then $\operatorname{ord} W^* = k-1$ and the term of the smallest degree in $W^*(z)$ is $c^* z^{k-1}$, where

$$c^* = a(k_2^* - k_1^*)a_{3-i,k_{3-i}} > 0, \tag{12}$$

We conclude that when a is small enough (depending on $\mathbf{q}$), the Wronskian W^* has one simple root in a neighborhood of each negative root of W, and in addition, one simple negative root close to zero, and a root of multiplicity $k-1$ at 0. To make this more precise, we denote the negative roots of W and W^* by

$$-x_{2d-2} < \ldots < -x_{k+1} \quad \text{and} \quad -y_{2d-2} < \ldots < -y_{k+1} < -y_k, \tag{13}$$

where $y_j = y_j(a)$. We have

$$y_j(0) = x_j, \quad \text{for} \quad 1 \leq j \leq n, \quad \text{and} \quad y_k(0) = 0. \tag{14}$$

Furthermore, if a is small enough (depending on $\mathbf{q}$)

$$a \mapsto y_k(a) \quad \text{is increasing and continuous.} \tag{15}$$

The set $w(k,\epsilon)$ is parametrized by a thorn $T = T(2d-2-k,\epsilon)$, where $\mathbf{x} = (x_{k+1},\ldots,x_{2d-2})$. There exists a continuous function $\delta : T \to \mathbf{R}_{>0}$, such that

$$\mathbf{q}^* \in b(\mathbf{k}^*), \quad \text{for} \quad a \in (0,\delta(\mathbf{x})), \quad \mathbf{x} \in T. \tag{16}$$

It remains to achieve (10) by modifying the thorn T. Consider the set

$$U^* = \{\mathbf{q}^* = F_a(\mathbf{q}_\mathbf{x}) : \mathbf{x} \in T,\, a \in (0,\delta(\mathbf{x}))\} \subset b(\mathbf{k}^*), \tag{17}$$

where $\mathbf{q}_\mathbf{x} \in U$ is some preimage under W of the polynomial (8) with $(x_{k+1},\ldots,x_{2d-2}) = \mathbf{x}$. Such preimage exists by assumption of Proposition 3 that the map $\mathbf{q} \mapsto W_\mathbf{q}$, $U \to w(k,T)$ is surjective. We apply Lemma 2 to the half-neighborhood (17) of T, with $x_k = a$, to obtain a thorn $T_1(2d-k-1,\epsilon_1)$. Then we apply Lemma 3 to the map $h : T_1 \to \mathbf{R}_{>0}^{2d-k-1}$, defined by $y_j = y_j(x_0,\mathbf{x})$, where y_j are as in (13), and $x_k = a$.

This map h satisfies all conditions of Lemma 3 in view of (14) and (15). This proves (10) and Proposition 3. □

We begin with the single element of $b(d-1,d)$ and apply operations F^i in some sequence, obeying the Rule above, while possible. As every step decreases k by 1, the total number of steps will be $2d-2$. We describe the sequence of steps by a sequence σ of 1's and 2's of length $2d-2$. The number i on the n-th place in this sequence indicates that operator F^i was applied on n-th step. The Rule above translates to the following characterization of all possible sequences σ:

a) The numbers of 1's and 2's in σ are equal.

b) In each initial segment of σ the number of 1's is not less than the number of 2's.

Such sequences are called *ballot sequences* (for two candidates), see, for example, [20]. The number of ballot sequences of length $2d-2$ is the Catalan number u_d.

Applying $2d-2$ times Proposition 3 according to each ballot sequence σ we obtain in the end an open subset $U_\sigma \subset b(0,1)$ which is mapped surjectively by the Wronski map onto $w(0,T)$ for some thorn T. As the intersection of any finite set of thorns of the same dimension contains a thorn of the same dimension by Lemma 1, we may assume that this set $w(0,T)$ is the same for all sequences σ.

When applying Proposition 3 we can make the range of parameter a as small as desired; using this we can assure that the sequence of coefficients of the pair (6) is monotone: the coefficients decrease in the order of their appearance. This implies that the sets U_σ with different σ are disjoint.

So we obtain u_d disjoint open sets U_σ, and each of them is mapped onto $w(0,T)$ continuously and surjectively by the Wronski map W. As the number of preimages of any point under W is at most u_d we conclude that the maps $W : U_\sigma \to w(0,T)$ are homeomorphisms for all σ.

Thus each point of the open subset $w(0,T) \subset X^{2d-2}$ has exactly u_d preimages under W and all these preimages are real. Each of these preimages corresponds to an analytic branch of the inverse W^{-1} on $w(0,T)$. The branches are enumerated by ballot sequences.

4 Completion of the proof

Let us fix a thorn T such that each polynomial in $w(0,T)$ has u_d different real preimages under the Wronski map, as in the end of the previous section.

To each of these preimages $\mathbf{q} = (q_1, q_2)$ corresponds a rational function $r(\mathbf{q}) = q_1/q_2$ in R^d with $2d-2$ distinct real critical points which has a net $\gamma(\mathbf{q})$. We take the rightmost critical point of these functions as distinguished vertices of the nets.

We claim that all these u_d nets are different. To prove the claim we just show how to determine the net from the ballot sequence and vice versa.

Proposition 4. *Let $k = 0$, and let p be a polynomial in $w(0, T)$ of the form (8). Let $\mathbf{q} = (q_1, q_2)$ be a polynomial pair as in (6) corresponding to a point in $W^{-1}(p)$, with the ballot sequence σ, and $g = r(\mathbf{q}) = q_1/q_2$.*

Then the net of g contains an edge between x_m and x_{m+1} if and only if the m-th member of the sequence σ is 1.

Proof. It is enough to investigate what happens to a net when an operator F^i of Proposition 3 is applied. We see from (6) and (9) that the degree of q_1/q_2 increases if and only if $i = 1$. Corollary 2 of Proposition 2 says that this happens if and only if the net has an edge between x_m and x_{m+1}. This proves Proposition 4. □

So we obtained a polynomial $p_0 \in w(0, T)$ of degree $2d - 2$ with $2d - 2$ real roots whose preimage under the Wronski map consists of u_d pairs with different nets. Each of these preimages corresponds to a holomorphic inverse branch of the Wronski map in $w(0, T)$. Let p_1 be any real polynomial of degree $2d - 2$ with $2d - 2$ real roots. Then there exists a path $p_t : t \in [0, 1]$ in X^{2d-2} connecting p_0 and p_1, such that all p_t are polynomials with $2d - 2$ distinct roots. For example one can connect the corresponding roots of p_0 and p_1 linearly. We do analytic continuation of all the inverse branches of the Wronski map along this path. As critical points of our rational functions cannot collide (because the zeros of their Wronskians p_t do not collide), their nets do not change under the continuation. Suppose that this analytic continuation to $t = 1$ is impossible. Let t_0 be the smallest singular point. Then p_{t_0} is a ramified value of the Wronski map and the full preimage $W^{-1}(p_{t_0})$ consists of fewer than u_d points. This full preimage still consists of real rational functions with all critical points real and distinct, so the nets are defined for all elements of this preimage. This means that at least two one-parametric families of rational functions with different nets tend to the same function with $2d-2$ distinct critical points, which is impossible by Corollary 1 of Proposition 2.

This proves theorems 1 and 2.

This proof clearly implies that for any given net there exists a unique class of real rational functions with all critical points real, and the critical points of these functions can be chosen arbitrarily, the result which was established in [1] with the help of the Uniformization theorem and rather complicated topological considerations.

To prove Theorem 3, we notice that 1-skeleton of every net of degree d can be obtained as the limit of 1-skeletons of nets with $2d-2$ vertices. So every net of degree d actually occurs as a net of a real rational function of the class of degree d with all critical points real. Counting the nets of degree d with q vertices of degrees $2a_1 \ldots, 2a_q$ gives the Kostka number $K_{\mathbf{a}}$ (see, for example, [6, Lemma 3]. So there are at least $K_{\mathbf{a}}$ classes of rational functions of degree d with prescribed real critical points. On the other hand, Schubert calculus [13] shows that there are at most $K_{\mathbf{a}}$ classes of rational functions with any prescribed critical points of multiplicities $a_1, \ldots, a_q$. This proves Theorem 3.

Corollary. *To each net of degree d corresponds exactly one class of real rational functions of degree d with prescribed real critical points.*

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