Houston Journal of Mathematics
© University of Houston
Volume , No. ,

NORMAL HOLOMORPHIC MAPS FROM $\mathbf{C}^*$ TO A PROJECTIVE SPACE

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Abstract. A theorem of A. Ostrowski describing meromorphic functions f such that the family $\{f(\lambda z) : \lambda \in \mathbf{C}^*\}$ is normal, is generalized to holomorphic maps from $\mathbf{C}^*$ to a projective space.

Let $f : \mathbf{C}^* \to \mathbf{P}^n$ be a holomorphic curve and

$$F = (g_0, g_1, \ldots, g_n) \tag{1}$$

some homogeneous representation of f. This means that g_j are analytic functions in $\mathbf{C}^*$ without common zeros. When $n = 1$, we have $\mathbf{P}^1 = \overline{\mathbf{C}}$, and f can be identified with a meromorphic function g_1/g_0 in $\mathbf{C}^*$.

There is a conformal Riemannian metric with the line element

$$|dz|/|z| \tag{2}$$

on $\mathbf{C}^*$ which is invariant with respect to conformal automorphisms $z \mapsto \lambda z,\ \lambda \neq 0$. The punctured plane with this metric is isometric to a cylinder of infinite length and circumference 2π.

A holomorphic curve in $\mathbf{C}^*$ is called *normal* if it is uniformly continuous with respect to this metric (2) and the Fubini-Study metric in $\mathbf{P}^n$. An equivalent property is that $\{z \mapsto f(\lambda z) : \lambda \in \mathbf{C}^*\}$ is a *normal family*: every sequence of these maps has a subsequence which converges uniformly on compacts with respect to the Fubini–Study metric to a holomorphic map $\mathbf{C}^* \to \mathbf{P}^n$.

A normal holomorphic curve in $\mathbf{C}^*$ has genus zero, which means that it has a *canonical* homogeneous representation

$$g_j(z) = A_j z^{m_j} \prod_{|z_{j,k}|<1} (1 - z_{j,k}/z) \prod_{|z_{j,k}|\geq 1} (1 - z/z_{j,k}), \tag{3}$$

2000 *Mathematics Subject Classification.* 30D45, 32A19.
Key words and phrases. Normal family, holomorphic curve.
Supported by NSF grant DMS-1067886.

where $z_{j,k}$ are the zeros of g_j, A_j is a constant, and m_j is an integer. We can and will always assume that $\min_j m_j = 0$, which defines these integers uniquely. It is clear that $z_{j,k}$ are uniquely defined by f, and A_j are defined up to a common multiple.

In fact,

$$A_j = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g_j(e^{i\theta})|d\theta\right).$$

A. Ostrowski [4] considered normal meromorphic functions ($n = 1$), and completely characterized them in terms of parameters of the canonical representation, see also [3, Ch. VI], for an exposition of this work.

In this paper, the result of Ostrowski is extended to arbitrary dimension n.

We begin with a reformulation of convergence of curves in terms of homogeneous coordinates. Consider a sequence $F_k = (g_{k,0}, \dots, g_{k,n})$ of $(n+1)$-tuples of holomorphic functions in an arbitrary region D. We assume that coordinates of each F_k have no zeros common to all of them. Then we have a sequence of holomorphic curves $f_k : D \to \mathbf{P}^n$.

Lemma 1. *The sequence (f_k) converges with respect to the Fubini–Study metric, uniformly on compacts in D, if and only if the following two conditions are satisfied:*
(i) For every compact $K \subset D$, there exist functions h_k holomorphic on K, having no zeros, and such that for every j there exists a limit uniform on K, with respect to the Euclidean metric in $\mathbf{C}$:

$$\lim_{k\to\infty} h_k g_{k,j} = g_{\infty,j}, \tag{4}$$

and
(ii) If (4) is satisfied with some h_k as in (i) then functions $g_{\infty,0}, \dots, g_{\infty,n}$ have no common zeros.

Some, but not all, functions $g_{\infty,j}$ may be identically equal to 0. Condition (ii) means that the rest of them have no common zeros.

Proof. It is evident that (i) and (ii) imply convergence of (f_k).

In the other direction, let $f_k \to f$ converge. Let $F = (g_0, \dots, g_n)$ be a homogeneous representation of f. Let $I \subset \{0, \dots, n\}$ be the set of indices for which $g_j \not\equiv 0$. There exist $\delta > 0$ and a finite open covering $\{D_j : j \in I\}$ of K such that $|g_j(z)| \geq \delta,\ z \in D_j,\ j \in I$. This implies that Fubini–Study distance from $f(D_j)$ to the hyperplane $w_j = 0$ is positive, so $g_{k,j}$ are free from zeros on D_j when k is large enough.

On D_j we define $h_{k,j} = g_j/g_{k,j}$. These functions are holomorphic and zero-free on D_j when k is large enough. We have $g_{k,i}/g_{k,j} \to g_i/g_j$ uniformly on $D_i \cap D_j$. Let $p_{k,i,j} = h_{k,i}/h_{k,j}$ on $D_i \cap D_j$. Then

$$p_{k,i,j} \to 1 \tag{5}$$

uniformly on $D_i \cap D_j$ and we have the cocycle condition

$$p_{k,i,j}p_{k,j,l}p_{k,l,i} = 1 \tag{6}$$

on triple intersections. In view of (5) we can define the principal branches of $\log p_{k,i,j}$ on the double intersections. Then there exist holomorphic functions $\phi_{k,j}$ on D_j such that $\log p_{k,i,j} = \phi_{k,i} - \phi_{k,j}$ on $D_i \cap D_j$, and $\phi_{k,j} \to 0$ as $k \to \infty$, on D_j (we may need to shrink the D_j a little at this step). See [2], theorems 1.2.2, $1.4.3', 1.4.4$ and 4.4.2. Now we set $h_k = h_{k,j}\exp(-\phi_{k,j})$ and these h_k do the job. This proves (i).

The functions h_k we constructed have property (ii). Now we show that (ii) must hold for every sequence of functions h_k as in (i). Let $I' = \{j : g_{\infty,j} \equiv 0\}$, and $I = \{0, \ldots, n\}\backslash I'$. Suppose that z_0 is a common zero of $g_{\infty,j},\ j \in I$. Then for every $\epsilon > 0$ there is a closed disc G centered at z_0 such that for $j \in I$ and k large enough, each function $h_k g_{j,k}$ has a zero in G. Let

$$M_k = \max\{|h_k g_{k,j}(z)| : z \in G, j \in I\}.$$

Then M_k is bounded from below as $k \to \infty$ by a constant that depends only on G. So we have

$$\max\{|h_k g_{k,j}(z)| : z \in G, j \in I'\} = o(M_k).$$

This means that for some points $z_k \in G$, $f_k(z_k)$ tends to the subspace

$$H_{I'} = \{(w_0, \ldots, w_n) : w_j = 0, j \in I'\}.$$

On the other hand, $f_k(z),\ z \in G$ visits every hyperplane H_j defined by $w_j = 0$ for $j \in I$. As these hyperplanes and subspace $H_{I'}$ have empty intersection, diameter of $f_k(G)$ must be greater then a positive constant independent of k. This contradicts our assumption that G is a disc of radius ϵ which can be arbitrarily small, because the sequence (f_k) is equicontinuous. This completes the proof of Lemma 1.

For the future use, we need a restatement of condition (ii) which does not involve the limit functions $g_{\infty,j}$.

Suppose that (i) holds. Then (ii) is equivalent to the following condition:

There exist C and δ (depending on K and (f_k)) such that such that for every disc $D(z_0,\delta)$ with $z_0\in K$, and for every $I\subset\{0,\ldots,n\}$, whenever all $g_{k,j},\; j\in I$ have zeros in $D(z_0,\delta)$, we have

$$\max_{0\leq j\leq n}|h_kg_{k,j}(z_0)|\leq\max_{j\in I'}|h_kg_{k,j}(z_0)|+C, \tag{7}$$

where $I'=\{0,\ldots,n\}\backslash I$.

The equivalence of (ii) and (7), assuming that (i) holds, has been established in the proof of the second part of Lemma 1.

For a function g holomorphic in a ring $\{z: r_1<|z|<r_2\}$, we define

$$N(r,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(re^{i\theta})|d\theta,$$

and

$$\mathfrak{N}(t,g)=N(e^t,g).$$

It is well-known that this function $\mathfrak{N}$ is convex on (r_1,r_2), and piecewise-linear. It is linear (affine) on an interval (a,b) if g has no zeros in the ring $\{z: e^a<|z|<e^b\}$, and the derivative $\mathfrak{N}'$ has a jump k at the point t if g has k zeros on the circle $|z|=e^t$. All this follows from the Jensen formula.

Lemma 2. *Using the notation of Lemma 1, suppose that F_k are defined in a ring $\{r_1<|z|<r_2\}$ and that (f_k) converges to a limit, so that (i) and (ii) hold. Then there exist linear functions $\ell_k(t)=a_kt+b_k$, such that for every $j\in[0,n]$ the limit*

$$\lim_{k\to\infty}\mathfrak{N}(t,g_{k,j}-\ell_k)<+\infty$$

exists, possibly identically equal to $-\infty$, uniformly on every interval $[a,b]$ such that $\log r_1<a<b<\log r_2$.

Proof. Indeed, the functions h_k of Lemma 1 are zero-free in the ring, so $\mathfrak{N}(t,h_k)$ are linear functions.

Let us fix some interval $(-a,a)$ and consider $(n+1)$-tuples

$$\Phi=(\phi_0,\phi_1,\ldots,\phi_n)$$

of convex functions on $(-a,a)$. We say that a sequence Φ_k of such tuples *converges uniformly* if for each j the coordinates $\phi_{k,j}$ converge uniformly on compact subintervals to finite convex functions, or to identical $-\infty$. A family of such $(n+1)$-tuples of convex functions is called *normal* if every sequence contains a subsequence that converges uniformly. If $n=0$ and we are dealing with a family of convex functions, then the criterion of normality is that the family is uniformly bounded from above on each compact subinterval. The limit function is finite if in addition the functions of the sequence are bounded from below at some point. These statements are well-known and easy to prove.

An equivalent criterion of normality with all limit functions finite for $n = 0$ is that *all functions are bounded at some point, their derivatives are bounded at the same point, and the total jump of the derivatives is bounded on each compact subinterval.*

From this, it is easy to derive a criterion for every $n > 0$: *for normality of a family of* $(n+1)$*-tuples of convex functions, it is necessary and sufficient that the functions* $\phi = \max_j \phi_j$ *form a normal family with* $n = 0$.

Lemma 3. *Let* $X = \{\Phi\}$ *be a set of* $(n+1)$*-tuples of convex functions on* $(-a, a)$. *Suppose that there exist linear functions* ℓ_Φ *such that the family*

$$\{\Phi - \ell_\Phi\} = \{(\phi_0 - \ell_\Phi, \dots, \phi_n - \ell_\Phi)\}$$

is normal on $(-a, a)$. *Then one can take*

$$\ell_\Phi(t) = \phi(0) + \phi'(0)t, \quad \textit{where} \quad \phi = \max_j\{\phi_0, \dots, \phi_n\},$$

and ϕ' *is the derivative from the right.*

This is an immediate consequence from what was said before the lemma.

Now we return to our original setting: f is a normal holomorphic map from $\mathbf{C}^*$ to $\mathbf{P}^n$, and F is a canonical representation of f as in (1), (3).

We are going to state two conditions for the curve f to be normal. We define

$$\mathfrak{N}(t, F) = \max_{j=0}^{n} \mathfrak{N}(t, g_j).$$

The derivative $\mathfrak{N}'(t, F)$ is always understood as the derivative from the right, so it takes only integer values, because $\mathfrak{N}'(t, g_j)$ takes only integer values.

Our first necessary condition of normality is a consequence of Lemma 3:

For every $a > 0$ *there exists* $C(a) > 0$ *such that*

$$\mathfrak{N}(t, F) - \mathfrak{N}(s, F) - \mathfrak{N}'(s, F)(t - s) \le C(a), \quad |t - s| < a. \tag{8}$$

Condition (8) is equivalent to

$$\mathfrak{N}(t, F) - \mathfrak{N}(s, F) - \mathfrak{N}'(s, F)(t - s) \le C_1(1 + (t - s)^2), \tag{9}$$

for some $C_1 > 0$ and all s, t.

Our *second condition* is related to statement (ii) of Lemma 1 and (7):

There exists $\delta > 0$ and $C > 0$ such that for every disc (with respect to the metric (2)) of radius δ, centered at a point $w \in \mathbf{C}^*$, the following condition is satisfied: if the disc contains zeros of functions g_j for $j \in I \subset \{0, \ldots, n\}$ then

$$N(|w|, F) \leq \max_{j \in I'} N(|w|, g_j) + C, \tag{10}$$

where I' is the complement of I in $\{0, \ldots, n\}$.

We will later prove that this condition is necessary for normality of f.

These two conditions of normality of a curve f are formulated in terms of parameters of formula (3) for the homogeneous coordinates. We give explicit expressions of the functions $N(r, g_j)$ in terms of these parameters:

$$N(r, g_j) = \log |A_j| + m_j \log r + \int_{[1,r]} n(1, t, g_j) \frac{dt}{t}.$$

Here $n(r_1, r_2, g_j)$ is the number of zeros of g_j in the ring $r_1 < |z| \leq r_2$.

Now we prove

Theorem 1. *Conditions (8) and (10) are necessary and sufficient for normality of a curve f with homogeneous representation (1), (3).*

Necessity of condition (8) has already been proved in lemmas 2 and 3. To prove the rest we use the following

Lemma 4. *Let (f_k) be a sequence of holomorphic curves, $f_k(z) = f(\lambda_k z)$, where f has a homogeneous representation (1), (3). Suppose that condition (8) with $s = 0$ holds for these curves uniformly with respect to k, and set*

$$F_k(z) = (g_0(\lambda_k z), \ldots, g_n(\lambda_k z)) = (g_{k,0}, \ldots, g_{k,n}),$$

and

$$h_k(z) = \exp(-\mathfrak{N}(0, F_k)) z^{-\mathfrak{N}'(0, F_k)}. \tag{11}$$

Choose a subsequence on which $\mathfrak{N}(t, h_k g_{k,j})$ tend to limits as $k \to \infty$, and let I be the set of indices j for which the limit is finite, and I' is the rest of the indices.

Then:

a) $h_k g_{j,k}$ tend to limits, not identically equal to zero, for $j \in I$, and
b) $h_k g_{j,k}$ tend to zero for $j \in I'$.

Proof of the lemma. Condition (9) implies that

$$\mathfrak{N}(t, F_k) \leq C_1(1 + t^2), \tag{12}$$

for all k, t and some $C_1 > 0$.

We first prove a). We define functions h_k by (11). These functions are holomorphic and zero-free because $\mathfrak{N}'(t,F)$ are integers. For $j \in I$, the functions

$$\mathfrak{N}(t, h_k g_{k,j}) = \mathfrak{N}(t, g_{k,j}) - \mathfrak{N}(0, f_k) - \mathfrak{N}'(0, f_k)t$$

are convex, uniformly bounded on any interval, the jumps of their derivatives are integers, so the total jump of the derivatives is bounded on every interval. Moreover, this total jump is at most a constant times the length of the interval, so we conclude that the $g_{k,j}$ have at most $C|b-a|$ zeros on every interval $[a,b]$, so one can pass to the limit in formulas (3), after multiplication of these formulas by the h_k. So the coordinates with $j \in I$ tend to non-zero limits, after choosing a subsequence.

Now we prove b), that is that remaining coordinates tend to zero. We fix some $j \in I'$ and will omit it from the formulas, because all argument applies to any such coordinate. We will also omit the index k to simplify our formulas. So $hg = h_k g_{k,j}$. We are going to prove that

$$B(x) = \log \max_{\theta} |(hg)(e^{x+i\theta})| \to -\infty,$$

uniformly for $|x| \leq \log 2$. We assume for simplicity of formulas that $x = 0$. We represent hg as a canonical product of the form (3), and denote by $n(r)$ the number of zeros on the in the ring $\{z : 1 \leq |z| \leq r\}$ if $r > 1$ or $\{z : r < |z| < 1\}$ if $r < 1$, and $\mathfrak{n}(t) = n(e^t)$. Then we have

$$B(0) \leq \log |A| + \int_1^\infty \log\left(1 + \frac{1}{\xi}\right) dn(\xi) - \int_0^1 \log(1+\xi) dn(\xi).$$

Integrating by parts we obtain

$$B(0) \leq \log |A| + \int_1^\infty \frac{n(\xi)d\xi}{\xi(1+\xi)} + \int_0^1 \frac{n(\xi)d\xi}{(1+\xi)}.$$

Changing the variable $\xi = e^t$ gives

$$B(0) \leq \log |A| + \int_{-\infty}^\infty \frac{\mathfrak{n}(t)dt}{1+e^{|t|}}. \tag{13}$$

Here A is the number from (3) which depend on $gh = g_k h_k$.

On the other hand,

$$\mathfrak{N}(x, hg) = \log |A| + mx + \int_{[0,x]} \mathfrak{n}(t)dt.$$

By integrating (13) by parts once more, we obtain

$$B(0) \leq \int_0^\infty (\mathfrak{N}(t, hg) + \mathfrak{N}(-t, hg)) \frac{e^t}{(1+e^t)^2} dt. \tag{14}$$

Now $\mathfrak{N}(0,hg)=\log|A|$, $\mathfrak{N}$ is convex and $\mathfrak{N}(t,hg)\leq C_1(1+t^2)$ in view of (12). These conditions imply that

$$\mathfrak{N}(t,hg)\leq 2(C_1+\sqrt{-C_1\log|A|})|t|+\log|A|.$$

Substituting this inequality to (14), we obtain $B(0)\leq\log|A|+C_2\sqrt{-\log|A|}+C_3$. As $A=A_k\to 0$, this completes the proof of the lemma.

Now necessity of condition (10) follows immediately because (10) is now the same as (7): $N(|z_0|,h_kg_{k,j})=\log|h_kg_{k,j}(z_0)|+O(1)$ when $h_kg_{k,j}$ tends to a non-zero limit as $k\to\infty$.

Sufficiency also immediately follows from Lemmas 1 and 4. This completes the proof of the theorem.

Now we compare the result with Ostrowski's conditions. His conditions are:

a) The difference between the numbers of zeros and poles in any ring $r_1<|z|<r_2$ is bounded uniformly with respect to r_1,r_2.

b) The number of zeros of each coordinate in rings $r<|z|<2r$ is bounded,

c) The distance between a zero of g_0 and a zero of g_1 is bounded from below.
d) There is a constant C such that for each zero w of g_j, we have $N(|w|,g_j)\leq N(|w|,g_{1-j})+C,\ j=0,1.$

It is easy to see that our condition (10) with $n=1$ implies c) and d). When $|I|=2$, it is c) and when $|I|=1$, it is d).

Condition (8) with $n=1$ implies b). Condition a), which in our notation means that $|\mathfrak{N}'(t,g_0)-\mathfrak{N}'(t,g_1)|$ is bounded, is an easy consequence of (8) and (10), when $n=1$.

However, for $n\geq 2$, conditions a) and b) do not have to hold. Here is a simple example. Let

$$g_0(z)=\prod_{j=0}^{\infty}(1-2^{-n}z),\quad g_1(z)=\prod_{j=0}^{\infty}(1+2^{-n}z),$$

and take as g_2 any entire function with the property

$$T(r,g_2)=O(\log^{3/2}r),\quad r\to\infty.$$

It is easy to see that $\psi=f_0/f_1$ is a normal meromorphic function with all limits $\lim_{\lambda\to\infty}\psi(\lambda z)$ non-constant, and

$$\log|g_3(z)|\leq o(\max\{\log|g_0(z)|,\log|g_1(z)|\}).$$

These properties imply that the curve f with homogeneous coordinates (g_0, g_1, g_2) is normal, but g_2 can be chosen so that the number of its zeros in some rings $r < |z| < 2r$ is unbounded.

The author thanks Masaki Tsukamoto for his questions that stimulated this work and Sergei Favorov for a useful discussion.

References

[1] A. Eremenko, Normal holomorphic curves from parabolic regions to projective spaces, arXiv:0710.1281v1.
[2] L. Hörmander, An introduction to complex analysis in several variables, D. van Nostrand, Princeton, NJ, 1966.
[3] P. Montel, Leçons sur les families normales des fonctions analytiques et leurs applications, Paris, Gauthier-Villars, 1927.
[4] A. Ostrowski, Über Folgen analytischer Funktionen und einige Verschärfungen des Picardschen Satzes, Math. Zeitschrift, 24 (1925), 215–258.

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