Normal matrices

October 3, 2024

1. $\|Tx\| = \|T^*x\|$. This is not true for general matrices.

   Proof. We have

$$(Tx, Tx) = (x, T^*Tx) = (x, TT^*x) = (T^*x, T^*x).$$

2. If $Tv = \lambda v$ then $T^*v = \overline{\lambda}v$. This is not true for general matrices.

   Proof. $Tv = \lambda v$ implies $\|(T - \overline{\lambda}I)v\| = 0$. So $\|(T^* - \overline{\lambda}I)v\| = 0$, so $T^*v = \overline{\lambda}v$.

3. Eigenvectors of a normal matrix with distinct eigenvalues are orthogonal.

   Proof. Let $Tv = \lambda v$, $Tu = \mu u$. Then

$$\lambda(u, v) = (u, Tv) = (T^*u, v) = \mu(u, v).$$

If $\mu \neq \lambda$ then $(u, v) = 0$.