Real solutions of Painlevé VI and special pentagons

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Introduction

Richard Fuchs was a son of Lazarus Fuchs. Father Fuchs is famous for Fuchsian groups, and several (at least three different kinds of) "Fuchs conditions" in the analytic theory of differential equations.

Continuing the work of his father on linear ODE, Richard studied in 1905 the following differential equation[1]:

$$w'' - \left(\frac{1}{z-q} + \sum_{j=1}^{3} \frac{\kappa_j - 1}{z - t_j} \right) w' + \left(\frac{p}{z-q} - \sum_{j=1}^{3} \frac{h_j}{z-t_j} \right) w = 0$$

Figure 1: Richard Fuchs

with 5 singularities at

$$(t_1, t_2, t_3, t_4, q) := (0, 1, x, \infty, q).$$

[1]His motivation probably was the Riemann problem for the simplest case not covered by Riemann's work: equation with four regular singularities (Heun's equation). Parameter count due to Poincaré shows that this Riemann problem cannot have a solution unless one introduces one apparent singularity. This is how the equation with five singularities, one of them apparent, arises.

The singularities at t_j have exponents $\{0, \kappa_j\}$, for $1 \leq j \leq 3$, and the exponents at q are $\{0, 2\}$. R. Fuchs imposed the following conditions:

a) the singularity at ∞ is regular, and has exponent difference κ_4, and

b) the singularity at q is apparent (has trivial monodromy).

For given $\kappa_j,\ 1 \leq j \leq 4$, and given $p, q, x = t_3$, these conditions determine the rest of the parameters h_j uniquely.

Suppose that all κ_j are fixed, and let us move x continuously. R. Fuchs asked the question: *how should $p(x), q(x)$ change so that the monodromy of this equation remains unchanged?* By a simple calculation he obtained the answer: q must satisfy the following non-linear ODE:

$$\begin{aligned} q_{xx} &= \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-x}\right) q_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{q-x}\right) q_x \\ &+ \frac{q(q-1)(q-x)}{2x^2(x-1)^2}\left\{\kappa_4^2 - \kappa_1^2 \frac{x}{q^2} + \kappa_2^2 \frac{x-1}{(q-1)^2} + (1-\kappa_3^2)\frac{x(x-1)}{(q-x)^2}\right\}, \end{aligned}$$

The equation for $p(x)$ will be mentioned later. This was the first example of what is called an "isomonodromic deformation" today.

At approximately the same time this non-linear equation appeared in a different problem. As we know, solutions of linear ODE are analytic in the same region (at least) where the coefficients are analytic. But for non-linear equations this is not so: singularities of solutions may appear anywhere, and they depend on the initial conditions. For example, equation $2yy' = 1$ has general solution $y(z) = \sqrt{z-c}$ with a singularity at c. E. Picard asked in 1889 for classification of non-linear equations whose solutions have no such "movable singularities", except movable poles. For first order ODE's the answer was known (Poincaré): there are two types of such equations, up to simple changes of the variables: the Riccati equation

$$y' = a(z)y^2 + b(z)y' + c(z),$$

and

$$(y')^2 = a(z)P(y),$$

where P is a polynomial with constant coefficients of degree at most 4. Picard also gave the first non-trivial example of a second order ODE without movable singularities: this was the equation above with $\kappa_j = 0,\ 1 \leq j \leq 4$.

PAUL PAINLEVÉ

BERTRAND GAMBIER

In the last decade of the 19th century Painlevé started to search for such equations systematically. In 1902 he obtained a classification of equations of the form

$$y'' = R(y', y, z),$$

where R is a rational function of y', y with coefficients meromorphic in z, whose solutions have no movable singularities. Most of the equations in this classification can be solved in terms of linear or first order equations. But some were new. Painlevé's initial classification had gaps, in particular he missed the equation written above. Classification was completed by Painlevé's student R. Gambier in 1909. He found 6 equations of the above form whose solutions have no movable singularities, and which cannot be reduced to linear or first order equations. These six equations are called Painlevé equations and denoted by PI–PVI. The equation above is PVI. Later Painlevé proved that for all these six equations, all solutions permit an unlimited analytic continuation in a fixed region which does not depend on the initial conditions. For PVI this region is $\mathbf{C}\backslash\{0,1\}$. The points $x = 0, 1, \infty$ are *fixed singularities* of PVI.

Away from the fixed singularities, the conditions of Cauchy's existence and uniqueness theorem are violated at the points where $q(x) \in \{0, 1, x, \infty\}$. These points x are removable singularities or poles of the solution. We call them *special points*.

In this talk we consider *real* solutions $q(x)$ of PVI with *real* parameters, on an interval of the real line between two adjacent fixed singularities $0, 1, \infty$. Without loss of generality we choose the interval $(1, \infty)$. We will explain a geometric interpretation of these solutions, and obtain an algorithm which determines the number and position of special points on the interval. More precisely, the outcome of the algorithm is a sequence of symbols $\{0, 1, x, \infty\}$ which shows the order in which the special points appear on $(1, \infty)$.

For example, a sequence $(0, 0, \infty, x)$ will mean that there are $1 < x_1 < x_2 < x_3 < x_4$ such that $q(x_1) = q(x_2) = 0$, $q(x_3) = \infty$, $q(x_4) = x_4$ and $q(x) \notin \{0, 1, x, \infty\}$ for all other $x > 1$.

How do we select a particular solution $q(x)$? There are the following methods:

a) one can solve the Cauchy problem with some non-special initial conditions $q(x_0) = q_0,\ q'(x_0) = q_0'$.

b) one can specify the second order linear equation with 5 singularities. When the PVI parameters κ_j and $x = x_0$ are fixed, the two remaining parameters $p = p_0$ and $q = q_0$ can serve as the initial conditions, but this is essentially the same as a). Indeed it was discovered by J. Malmquist that PVI is equivalent to the following Hamiltonian system

$$\frac{dq}{dx} = \frac{\partial h}{\partial p}, \quad \frac{dp}{dx} = -\frac{\partial h}{\partial q},$$

where the Hamiltonian $h(p, q, x)$ is nothing but the residue h_3 in the linear equation. The Hamiltonian is a second degree polynomial in p, so PVI is obtained by elimination of p from the system.

c) one can specify the monodromy representation corresponding to the linear equation. At least for generic values of PVI parameters and generic monodromies this specifies the linear equation (that is p_0 and q_0) uniquely.

We will use somewhat different method of assigning the initial conditions, and monodromy representation will be easily computed from our initial conditions. (Initial values of p_0, q_0 are difficult to compute directly from the monodromy). In fact we will study the isomonodromic deformation of the linear equation directly, without using PVI which only serves as the motivation.

Linear Fuchsian ODE with real parameters and circular polygons

Suppose that all parameters (singularities, exponents and accessory parameters) in a linear Fuchsian ODE

$$w'' + P(z)w' + Q(z)w = 0$$

are real. *Such equations will be called real.* Consider the ratio $f = w_1/w_2$ of two linearly independent solutions. This function is meromorphic in the upper half-plane H and is locally univalent there. Indeed,

$$f' = w_2^{-2}(w_1'w_2 - w_2'w_1),$$

and this is never 0, in $\mathbf{C}\backslash\{0, 1, x, q\}$, and f can have only simple poles in this region.

Consider an interval I between two singular points. Choosing real initial conditions at a point of I we obtain two real solutions, so f is real on I. For other initial conditions, f will be a linear-fractional transformation of the previous one, therefore it will map I into a circle.

From the local theory it follows that the singularity of f at a singular point t is the following:

$$f(z) = f(t) + (c + o(1))(z - t)^\alpha,$$

where α is the absolute value of the exponent difference at t. If $\alpha = 0$ but the singularity at t is not apparent, then

$$f(z) = f(t) + (c + o(1))/\log(z - t).$$

These formulas should be properly modified when $t = \infty$ or $f(t) = \infty$ by using local parameters. Such singularities are called *conical*. The number $\alpha \geq 0$ is called the *angle* at the singularity and we set $\alpha = 0$ when the asymptotic formula with log holds.

Notice that we measure all angles in half-turns instead of the radians!

Thus our function f is holomorphic in H, locally univalent in $\overline{H}\backslash\{t_j\}$, maps each interval (t_{j-1}, t_j) into some circle C_j, and has conical singularities at t_j.

Such functions are called *developing maps* (of circular polygons). The formal definition of a circular polygon is

$$Q = (\overline{D}, t_1, \dots, t_n, f),$$

where $\overline{D}$ is a closed disk, $t_j \in \partial D$ are distinct boundary points, and f is a developing map with conical singularities at t_j. The intervals (t_{j-1}, t_j) are called *sides*, the points t_j *corners* and the α_j are the interior angles at the corners.

Let us explain this definition informally. If f is injective in D then $f(D)$ is a subset of the Riemann sphere, this subset is what people usually call a circular polygon, a region on the sphere bounded by arcs of circles. The α_j are the interior angles of $f(D)$. In this simple case, there is no need in f and D to describe the polygon.

In the general case, the angles can be > 2, and the sides $f(t_{j-1}, t_j)$ can cover their circles more than once, so the picture of $f(D)$ becomes inadequate, as this image overlaps itself.

Instead one should think of a bordered *"Riemann surface $\overline{D}$ spread over the sphere"*, without ramification points inside, and such that the border projects into finitely many circles. It follows from our definition that this Riemann surface spread over the sphere has finitely many sheets.

Two circular n-gons $Q = (\overline{D}, t_1, \ldots, t_n, f)$ and $Q' = (\overline{D'}, t_1', \ldots, t_n', f_1)$ are *equal* if there is a conformal homeomorphism $\phi : \overline{D'} \to \overline{D}$ such that $\phi(t_j') = t_j$ and

$$f_1 = f \circ \phi. \tag{1}$$

In the special case when our polygons are subsets of the sphere, and all angles are different from 1, this means that they are equal as sets.

Two circular n-gons are called *equivalent* if instead of (1) we require only

$$f_1 = L \circ f \circ \phi,$$

with some linear-fractional transformation L. For polygons which are subsets of the sphere this means that one can be moved onto another by a linear-fractional transformation.

Let us continue our discussion of the real linear Fuchsian equation. We have seen that to each such equation one can associate a circular polygon with $D = H$. Now suppose that such a circular polygon with $D = H$ is given. Then the developing map f can be analytically continued along any path in $\overline{\mathbf{C}} \backslash \{t_1, \ldots, t_n\}$ by reflections. This defines a multi-valued meromorphic function whose Schwarzian

$$S_f(z) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is single-valued, because Schwarzian of f and of $L \circ f$ is the same for every linear-fractional transformation L.

The principal terms of S_f at the singularities t_j are determined from the angles α_j. It turns out that $R = S_f$ is a rational function with double poles, and f solves the Schwarz differential equation $S_f = R$. It is well known that every solution of such a differential equation is a ratio of two linearly independent solutions of a Fuchsian equation. This Fuchsian equation is not unique. However, if we normalize so that the exponents at the finite singularities are 0 and $\alpha_j \geq 0$, then the Fuchsian equation will be unique. There are other normalizations which give uniqueness, the most popular is $w'' + (1/2)S_f w = 0$.

To conclude: *There is a one-to-one correspondence between the equivalence classes of circular n-gons and normalized Fuchsian equations with all parameters real. The developing map defining a polygon is the ratio of two linearly independent solutions.*

Of course, this fact was well-known to Schwarz and possibly to Riemann.

ÉMILE PICARD

Special pentagons corresponding to the equation of R. Fuchs

This equation with five singularities defines a circular pentagon. But one singularity q is special: it has exponents $0, 2$ and trivial monodromy. Suppose that $q \in I = (t_{k-1}, t_k)$. Then f is meromorphic on I, maps I into a circle C, and has a single simple critical point at q. This means that f folds a neighborhood of q in I into an arc of C with one endpoint at $f(q)$.

We say in such case that our pentagon Q *has a slit*, and call $f(q)$ the *tip of the slit*. Pentagons of this type (with exactly one slit) will be called *special pentagons*. We assume for simplicity that the monodromy at each point $(t_1, t_2, t_3, t_4) = (0, 1, x, \infty)$ is non-trivial, that is no singularity except q is apparent.

There is a one-to-one correspondence between real normalized Fuchsian equations with 5 singularities, one of them apparent with exponent difference 2, and equivalence classes of special pentagons.

Examples of special pentagons which are subsets of the plane are shown in Figs. 6a,c and 7a,c,e,g. Symbols like (1) mean $f(1)$ etc. in parentheses indicate the f-preimages.

The sides of a special pentagon are mapped by f into 4 circles. Let C_j be the circle that contains $f([t_{j-1}, t_j])$. If $q \in (t_{k-1}, t_k)$ then two sides (t_{k-1}, q) and (q, t_k) are mapped into the same circle C_k. The intersections $C_j \cap C_{j+1}$ are not empty, they contain $f(t_j)$; and our assumption of non-trivial monodromy implies that $C_j \neq C_{j+1}$ for all $j \in \mathbf{Z}_4$.

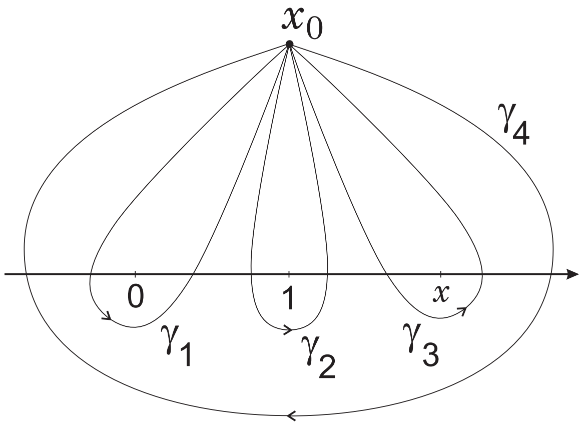

Figure 2: Loops defining L_1, L_2, L_3, and L_4.

For the description of monodromy, choose the starting point z_0 in H, and let γ_j be a simple positively oriented loop around t_j crossing the real line twice: first on the left of t_j and second time on the right, so that the interiors of the loops are disjoint. (The interior of the loop around ∞ contains ∞, of course). Such loops satisfy

$$\gamma_1\gamma_2\gamma_3\gamma_\infty = e,$$

where we use the usual convention that $\gamma_1\gamma_2$ is γ_1 followed by γ_2. Then the (projective) monodromy representation $\gamma \mapsto L_\gamma$, where L_γ is defined by $f^\gamma = L_\gamma \circ f$, and f^γ is the result of the analytic continuation of f along γ, is a homomorphism of the fundamental group into the group of linear-fractional transformations. In particular,

$$L_1 \circ L_2 \circ L_3 \circ L_\infty = \mathrm{id}.$$

The monodromy transformation L_j at t_j is obtained by the formula

$$L_j = \sigma_j\sigma_{j+1}, \tag{2}$$

where σ_j is the reflection in the circle C_j. Indeed to perform an analytic continuation along γ_j we first continue f analytically to the lower half-plane by reflection σ_j, and then continue back to H by reflection in the circle $\sigma_j C_{j+1}$. This last reflection is $\sigma_j\sigma_{j+1}\sigma_j$ and applying it after σ_j we obtain

$$\sigma_j\sigma_{j+1}\sigma_j\sigma_j = \sigma_j\sigma_{j+1}.$$

One can show that if a representation (2) of L_j in terms of σ_j exists, then the σ_j are unique, except in the case that all L_j commute. One can also write explicit conditions on the L_j which imply existence of representation (2).

Isomonodromic deformation in terms of special pentagons

We recall that the sides of a special pentagon are mapped by the developing map into four circles C_j, and these circles can be usually recovered from the projective monodromy representation.

How can a special pentagon be continuously deformed so that the four circles remain fixed? This is only possible by moving the tip of the slit along its circle.

One can easily show that for every special pentagon, one can move the tip of the slit by a small amount in both directions on its circle. The parameters x and q will be functions of the tip position on the circle, and one can show that x is strictly monotone as a function of the tip. So $q(x)$ is well defined locally and this is a solution of PVI. So we can denote our special pentagon by Q_x.

Now we look at the global picture. As the slit shortens, it will eventually vanish. At this moment q will collide with one or two corners. If it collides with two corners, then the limit of Q_x is a triangle, and this means that $x \to 1$ or $x \to \infty$. We say that Q_x *degenerates* if this happens.

Otherwise q collides with one corner, we have a limit quadrilateral, and $x \to x_0$, where x_0 is a special point. If $q \in (t_{k-1}, t_k)$, then vanishing of the slit at one corner means that q collides with either t_{k-1} or t_k.

When the slit lengthens, it eventually hits the boundary of Q_x from inside, and our special pentagon Q_x splits into two parts: the slit becomes a crosscut. If one of the parts into which it splits is a quadrilateral, we must have $x \to x_0 \in (1, \infty)$, and the limit of the developing map maps H onto this quadrilateral. Then x_0 is a special point. Otherwise $x \to 1$ or $x \to \infty$, and we do not have a limit quadrilateral: Q_x degenerates.

The question is what happens to Q_x when x passes beyond a special point x_0. There are four cases, three generic and one non-generic.

We explain only the three generic cases.

Transformation 1. Let $q \in (t_{k-1}, t_k)$, and as the slit vanishes, q collides with s which is either t_k or t_{k-1}. When $q = s$, we have a quadrilateral without a slit. As x passes s we must have a special pentagon with images of the sides on the same 4 circles, but q and s interchanged their order on ∂H, The slit which was on C_k is now on C_{k+1}, if $s = t_k$, and on C_{k-1} if $s = t_{k-1}$. See Fig. 3.

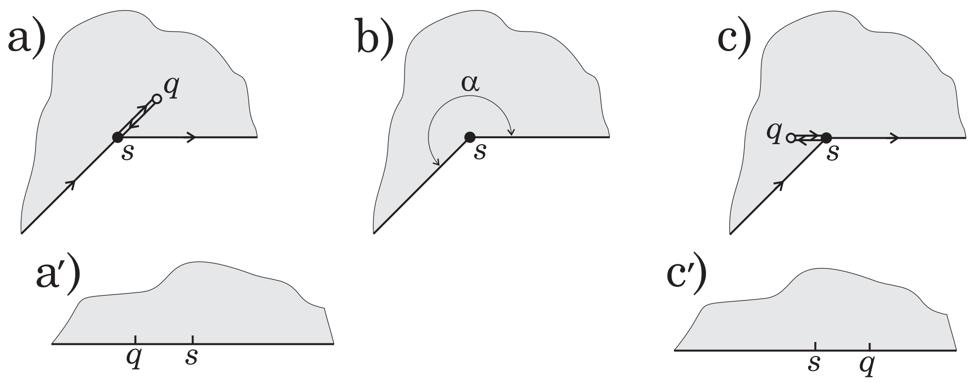

Figure 3: Transformation 1.

Transformation 2. Suppose that $q \in (t_{k-1}, t_k)$, and the slit lengthens. Then eventually it hits the boundary from inside of Q_x, and becomes a cross-cut. The cross-cut splits the pentagon into two parts. Let $s \in \partial H$ be the point where this collision happens (that is $f(q) \to f(s)$ as the slit lengthens).

We assume first that $s \notin \{t_j\}$, and that $s > t_k$ (wlog). If the interval $(q, s) \subset \partial H$ and the complementary interval $\partial H \backslash (q, s)$ each contain two of the t_j, then the quadrilateral degenerates, so $x \to 1$ or $x \to \infty$. A special point will correspond to the case when one of the intervals (q, s) or its complement contains one of the t_j, which we denote by t, and the other one three of them. We assume that the angle α at t satisfies $\alpha \neq 0$. Suppose that $q < t < s$ and the other three t_j are outside (q, s). These three points t_j serve as the normalization points of f, and when $f(q)$ tends to $f(s)$, the limit f will map H onto a quadrilateral with corners at s an those three t_j.

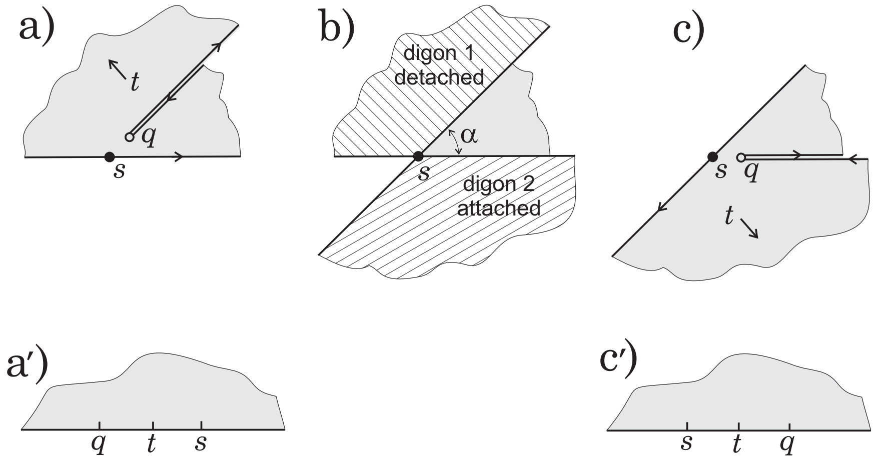

Figure 4: Transformation 2.

The part which splits away is a digon with corners at t and s. This digon is detached in the limit. In the z plane all three points q, t, s collide. Before this collision, it is a small neighborhood of t which is mapped on the would-be digon.

After the collision, we have $s < t < q$, and a new digon is attached. It is easy to see that this new digon has the same angle as the old one, and is bounded by the same two circles. We call it the "vertical digon" to the old one.

Transformation 3. When the slit lengthens, hits the boundary from inside, and the special pentagon splits, as in Transformation 2, we assume now that the slit hits a corner $s \in \{t_j\}$.

If $x \to x_0 \in (1, \infty)$, we have to obtain a non-degenerate quadrilateral in the limit, that is for $q \in (t_{k-1}, t_k)$ we must have $f(q) \to f(s)$, as the slit lengthens and splits the pentagon, where s is either t_{k-1} or t_k. This means that a disk (rather than digon) detaches in the limit, and a new disk is attached as x passes x_0, as shown in Fig. 5.

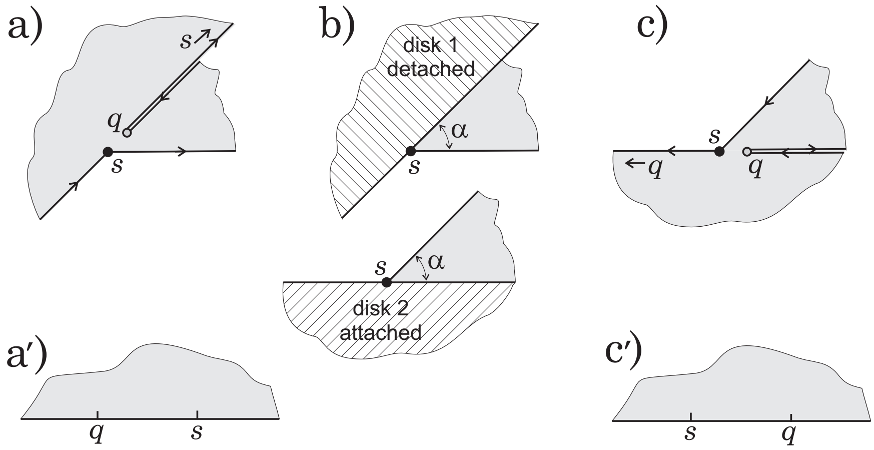

Figure 5: Transformation 3.

The non-generic cases occur only in special pentagons with some zero angles.

Examples

In these examples all polygons are subsets of the plane.

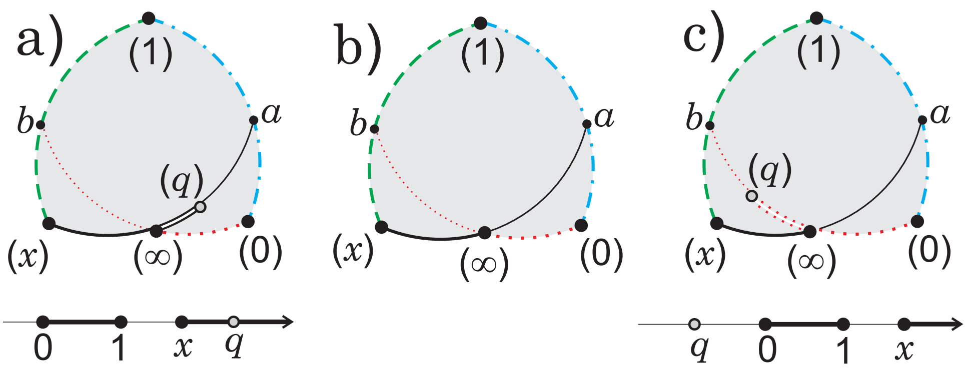

Figure 6: Global family in Example 1.

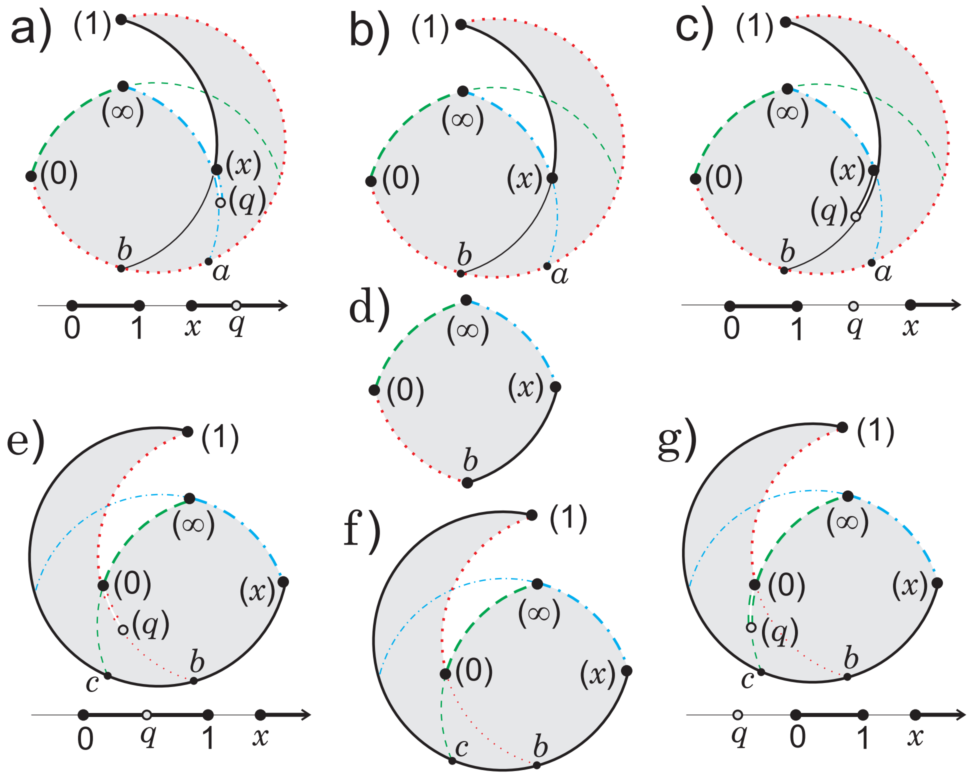

Figure 7: Global family in Example 2.

In Example 1, when the slit in a) lengthens and hits the boundary, we have $x \to 1$. As the slit in a) shortens, x increases. When the slit vanishes we obtain the limit quadrilateral in picture b); at this point $q(x) = \infty$. Then the new slit grows as in picture c) and when it hits the boundary, $x \to +\infty$. Therefore the solution $q(x)$ represented in Fig. 6 has only one special point on $(1, +\infty)$, and it is a pole. We had one transformation of type 1 in this example.

In Example 2, when the slit in a) lengthens, x decreases and $x \to 1$ as the slit hits the boundary. When the slit in a) shortens, x increases. Then we have transformation 1 in b), transformation 2 in d) and transformation 1 in f). The solution represented in Fig. 6 has 3 special points: $x_0 < x_1 < x_2$ with $x_0 > 1$, $q(x_0) = x_0$, $q(x_1) = 1$, $q(x_2) = 0$.

Representation of polygons by nets

In general, our slit pentagons are not subsets of the plane, so we need a tool to describe them.

Let $Q = (\overline{D}, t_1, \ldots, t_n, f)$ be a circular n-gon. As explained above, it defines the n circles C_j, namely C_j is the circle containing $f((t_{j-1}, t_j))$. These circles are not necessarily distinct, but C_j and C_{j+1} are distinct and have non-empty intersection. Circles C_j define a cell decomposition of the sphere which we call the *lower configuration*. The f-preimage of the lower configuration is a cell decomposition of the closed disk $\overline{D}$ which is called the *net* of our polygon. Vertices of the net at the corners are labeled by t_j. Two nets are considered the same if there is an orientation-preserving homeomorphism of $\overline{D}$ sending one net to another and labeled vertices to similarly labeled vertices.

To ensure that a net represents a unique polygon, one needs some normalization. For example, let e be the boundary edge of the net that follows t_1, and T the unique 2-cell whose boundary contains e. Then specifying the cells

$$(f(t_1),\ f(e),\ f(T))$$

of the lower configuration will define the polygon uniquely.

So a polygon is completely determined by the lower configuration, the net and the normalization data.

It is difficult to describe intrinsically all possible nets on a given lower configuration. But in the case when $n = 4$ and the lower configuration is homeomorphic to a generic quadruple of great circles, one can give such an intrinsic description. A generic quadruple of great circles is shown in Fig. 8.

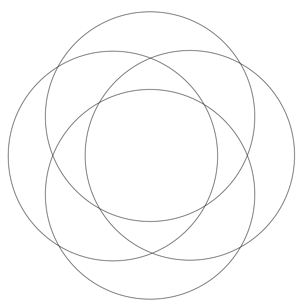

Figure 8: Four generic great circles

The corresponding cell decomposition of the sphere has the following property:

a) *any pair of* 2*-cells adjacent along a* 1*-cell consists of a triangle and a quadrilateral.*

This property is inherited by the net. Two additional property of the net are:

b) *every interior vertex has degree* 4*, and every vertex on a side has degree 3*, and

c) *the degrees of the corners (as vertices of the net) are even.*

The last property follows from our assumption that the circles C_j and C_{j+1} are distinct.

One can show that these three properties a), b) and c) characterize the nets over lower configurations homeomorphic to generic configurations of 4 great circles.

This permits to construct many examples of nets, circular quadrilaterals and special circular pentagons. In fact a complete classification of all possible nets over the lower configuration in Fig. 8 is available.

Transformations 1, 2, 3 above can be explicitly performed on the nets.

Lower configurations of four great circles correspond to $PSU(2)$ monodromy representations.

Properties of special points of real PVI solutions strongly depend on the topological type of the lower configuration. For example, the number of special points can be infinite only if some two circles of the lower configuration are disjoint. We conjecture that this condition is also sufficient for a PVI solution to have infinitely many real special points.