Oscillation of Fourier Integrals with a spectral gap

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Abstract

Suppose that for a real function f on the real line, the support of its Fourier transform is disjoint from an interval $(-a, a)$. We prove the conjecture of B. Logan that under these assumptions the lower density of the sequence of sign changes of f is at least a/π. This result assumes a growth condition: f should be integrable with respect to a non-quasianalytic weight. We show that this growth restriction is best possible (for sufficiently regular weights). For functions f of faster growth, Logan's conjecture does not hold, and for this class we prove a weaker result estimating the number of sign changes from below in terms of an averaged lower density.

Keywords: Spectral gap, Sign changes, Entire functions, Heat equation.

Supposons que le support de la transformée de Fourier d'une fonction réelle f sur la droite réelle est disjoint d'un intervalle $(-a, a)$. Nous démontrons la conjecture de B. Logan et al. que, sous ces hypothèses, la densité inférieure de la suite des changements de signe de f est au moins a/π. Ce résultat suppose une condition de croissance: f doit être intégrable par rapport à un poids non quasi-analytique. Cette restriction de croissance est la meilleur possible, même pour des fonctions f à spectre borné. Pour des fonctions et distributions f de croissance plus rapide, nous montrons un résultat plus faible qui

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donne une estimation inférieure du nombre de changements de signe en fonction d'une densité inférieure moyénnée.

Mots-clés: Trou spectral, Changements de signe, Fonctions entières, Équation de la chaleur.

1 Introduction

Suppose that in a real Fourier series, the first m terms vanish:

$$f(x) = \sum_{n \geq m} (c_n e^{inx} + \overline{c}_n e^{-inx}), \quad f \neq 0. \tag{1}$$

Then f has at least $2m$ changes of sign on the interval $|x| \leq \pi$. For trigonometric polynomials this follows from a result of Sturm [40]; the general case is due to Hurwitz.

Here is a simple proof. The number of sign changes is even. If f has at most $2(m-1)$ changes of sign then we can find a trigonometric polynomial g of degree at most $m-1$ that changes sign at the same places as f. Then fg is of constant sign which contradicts the orthogonality of f and g.

We consider the following extension of this result to Fourier integrals.

Statement 1 *Suppose that a real function f has a spectral gap, that is its Fourier transform is zero on an interval $(-a, a)$. Then the number of sign changes $s(r, f)$ of f on the interval $(0, r)$ satisfies*

$$\liminf_{r \to \infty} \frac{s(r, f)}{r} \geq \frac{a}{\pi}. \tag{2}$$

For example, if $f(x) = \cos x$, the spectral gap is $(-1, 1)$, and $s(r, f) = r/\pi + O(1)$, so equality holds in (2). It is essential in Statement 1 that the spectral gap contains zero: the function $\cos x + 1$ never changes sign.

The proof of Hurwitz's theorem given above can be generalized to estimate from below the *maximal* density of sign changes of f (see [9] for its definition) but apparently it cannot be used to estimate the *lower* density (2).

Statement 1 was conjectured by B. Logan in 1965 [27], and his conjecture was recently repeated in several places, for example, in [1, 38]. In the next section we give a survey of known results. In this paper we prove Statement 1 for a class of measures on the real line whose spectra can be defined as supports of their Fourier transforms in the sense of distributions.

A "measure" in this paper means a Radon measure, that is a continuous real linear functional on the space of real continuous functions with compact support. A measure f on the real line has a canonical decomposition into positive measures $f = f^+ - f^-$, and we denote by $|f| = f^+ + f^-$ the *variation* of f. A locally integrable function f will be identified with the measure $f dx$. Moreover, we will use the notation

$$\int f(x)dx$$

for the integral over the real line, even when the measure f is not absolutely continuous. We don't write the limits of integration for integrals over the real line. The word "positive" means "non-negative".

The *number of sign changes* of a measure f on an open interval I will be denoted by $s(I, f)$. It is defined as the minimal degree of a polynomial p such that the restriction of pf on I is a positive measure. If $I = (0, r)$, $r \in \mathbf{R}$, we use the notation $s(r, f) = s(I, f)$. The same definition of the number of sign changes is applicable to distributions, provided that restrictions on intervals make sense, for example, to distributions of Schwartz's space $\mathscr{D}'$.

We use a general definition of a spectrum which is due to Carleman [11]. Let f be a measure that satisfies

$$\int e^{-\lambda|x|}|f(x)|dx < \infty \quad \text{for all} \quad \lambda > 0. \tag{3}$$

Then the functions

$$F^+(\zeta) = \int_{-\infty}^{0-} e^{-ix\zeta} f(x)dx \quad \text{and} \quad F^-(\zeta) = -\int_{0-}^{\infty} e^{-ix\zeta} f(x)dx \tag{4}$$

are analytic in the upper and lower half-planes, respectively, and the *generalized Fourier transform* is defined as a *hyperfunction*, that is a pair (F^+, F^-) of analytic functions in the upper and lower half-planes, modulo addition of an entire function to both F^+ and F^-. The Fourier transform in the usual sense, if it exists, is recovered as the difference of the boundary values, $\hat{f}(t) = F^+(t) - F^-(t)$, $t \in \mathbf{R}$. This suggests a general definition of the spectrum of a measure satisfying (3), see, for example, [5].

Definition 1 *The spectrum of a measure satisfying* (3) *is the complement of the maximal open set* $U \subset \mathbf{R} \cup \{\infty\}$ *such that* F^+ *and* F^- *are analytic continuations of each other through* U.

Thus a measure f has a spectral gap $(-a, a)$ if F^+ and F^- are analytic continuations of each other through the interval $(-a, a)$. If $f \in L^1 := L^1(\mathbf{R})$, and the Fourier transform $\hat{f}$ is defined in the classical sense, then the presence of a spectral gap $(-a, a)$ in the sense of Definition 1 is equivalent to the condition $\hat{f}(t) = 0$ for $t \in (-a, a)$. The same is true if f is a tempered measure whose Fourier transform is defined in the sense of Schwartz's tempered distributions, see [11].

In engineering literature, functions with a spectral gap are called high-pass signals.

Condition (3) is too weak to develop a proper extension of Harmonic Analysis [5]. For example, it may happen for a locally integrable function $f \neq 0$ that F^+ and F^- are restrictions of a single entire function, so according to our Definition 1, the spectrum of f consists of the point ∞.

Following Beurling [5, 6], we consider a stronger condition

$$\int e^{-\lambda\omega(x)}|f(x)|dx < \infty \quad \text{for some} \quad \lambda > 0, \tag{5}$$

where $\omega \geq 0$ is a real function, called a *weight*, with the property

$$\int \frac{\omega(x)}{1+x^2}dx < \infty, \tag{6}$$

and subject to some regularity conditions. For locally integrable functions f, conditions (5) and (6) imply

$$\int \frac{\log^+|f(x)|}{1+x^2}dx < \infty. \tag{7}$$

A function $\omega \geq 0$ on the real line will be called a *Beurling–Malliavin weight* (BMW) if it satisfies (6) and, in addition, has at least one of the following properties:

(i) ω is uniformly continuous, or

(ii) $\exp \omega$ is the restriction of an entire function of exponential type to the real line.

Notice that for a BMW ω, (5) implies (3), so Definition 1 applies to measures satisfying (5) with a BMW ω. The main result of this paper is

Theorem 1 *Let ω be a BMW. If $f \neq 0$ is a measure satisfying* (5) *and having a spectral gap $(-a, a)$, then* (2) *holds.*

In particular, Statement 1 is true for all bounded functions f. A version of Theorem 1 for Schwartz's tempered distributions can be derived using the arguments in [28]. All difficulties in the proof of Theorem 1 occur already in the case that f is a measure of finite variation on the real line; once it is proved for such measures an application of the Beurling–Malliavin Multiplier theorem gives the general case (see section 3). On the other hand, we will show in Example 2 below that the growth condition (5), (6) is best possible.

The theory of mean motion [19, 39] suggests a stronger version of (2):

$$\liminf_{x-y\to+\infty} \frac{s(x,f)-s(y,f)}{x-y} \geq \frac{a}{\pi}. \tag{8}$$

This is not true, even for bounded functions in L^1 that are restrictions to the real line of entire functions of exponential type:

Example 1 *For every pair of positive numbers $a < b$, there exists a real entire function f of exponential type b, whose restriction to the real line is bounded and belongs to L^1, which has a spectral gap $(-a, a)$, and the property that for a sequence of intervals $[y_k, x_k]$ whose lengths tend to infinity, f has no zeros on $[y_k, x_k]$.*

Examples of functions with a spectral gap and no sign changes on *one* long interval are contained in [27].

To show that condition (5) is essential in Theorem 1, we consider functions f with *bounded spectrum.* By definition, this means that the generalized Fourier transform (F^+, F^-) extends to a function F analytic in $\overline{\mathbf{C}}\backslash[-b, b]$, for some $b \geq 0$. Normalization condition $F(\infty) = 0$ defines F uniquely. A theorem of Pólya [9, 24] gives a precise description of such functions f: they are restrictions to the real line of entire functions of exponential type b, satisfying (3). Moreover $f \mapsto F$ is a bijection between the class of functions with spectrum on $[-b, b]$ and the class of functions F analytic in $\overline{\mathbf{C}}\backslash[-b, b]$ satisfying $F(\infty) = 0$. The inverse correspondence is given by

$$f(z) = -\frac{1}{2\pi}\int_\gamma F(\zeta)e^{iz\zeta}d\zeta, \tag{9}$$

where γ is a path going once around $[-b, b]$ counterclockwise.

In engineering literature functions with bounded spectrum are called band-limited signals. We abbreviate "entire function of exponential type" as *efet*. Functions with bounded spectrum that satisfy (7) form a subclass of efet called the *Cartwright class*.

Example 2 *For every sufficiently regular*[1] *weight* ω *with divergent integral* (6)*, and every positive numbers* $a < b$*, there exists a real efet* f *satisfying* $|f| \leq \exp \omega$*, whose spectrum is contained in* $[-b, -a] \cup [a, b]$*, and such that*

$$\liminf_{r\to\infty} s(r, f)/r < a/\pi.$$

We conclude that condition (6) is essential for validity of Statement 1. Convergence or divergence of the integrals (6) or (7) is a fundamental dichotomy in Harmonic Analysis, [6, 22].

For measures that satisfy only (3) we obtain weaker estimates than (2), in terms of the averaged densities

$$S(r, f) = \int_0^r \frac{s(t, f) + s(-t, f)}{t} dt \tag{10}$$

and

$$C(r, f) = \int_1^r \left(\frac{1}{t^2} + \frac{1}{r^2} \right) s(t, f) dt. \tag{11}$$

Theorem 2 *Let* f *be a measure satisfying* (3)*, having a spectral gap* $(-a, a)$*. Then*

$$\liminf_{r\to\infty} \frac{S(r, f)}{r} \geq \frac{2a}{\pi} \quad \textit{and} \quad \liminf_{r\to\infty} \frac{C(r, f)}{\log r} \geq \frac{a}{\pi}. \tag{12}$$

Consider, for example, those functions f satisfying (3) whose spectra consist of the single point ∞. This happens when $F^{\pm}$ are restrictions of an entire function F to the upper and lower half-planes. To obtain such example, take any entire function that satisfies $F(\zeta) = O(\zeta^{-2})$ as $\zeta \to \infty$ on the sets $\{\zeta : |\text{Im}\, \zeta| \geq \epsilon\}$, for every $\epsilon > 0$, and define f as the inverse Fourier–Carleman transform:

$$f(x) = -\frac{1}{2\pi} \int_\gamma F(\zeta) e^{ix\zeta} d\zeta,$$

where γ is the oriented boundary of the strip $\{\zeta : |\text{Im}\, \zeta| = \epsilon\}$, and $\epsilon > 0$. Our Theorem 2 implies that for such functions f, $S(r, f)/r \to \infty$.

The difference between Theorems 1 and 2 becomes especially transparent in the case that f has bounded spectrum. In the next section we sketch a

[1] The precise conditions needed are stated in section 11.

simple independent proof of both theorems for this case (Proposition 1). For functions with bounded spectrum, conditions (5) and (6) are equivalent to (7), and the proof of Proposition 1 clarifies the role of these conditions: (7) implies completely regular growth of f in the sense of Levin and Pfluger. This follows from a theorem of Cartwright and Levinson, [24]. In certain sense (7) is a minimal growth condition that implies completely regular growth. Without (7) the conclusion of Theorem 1 is not true anymore, but we have a weaker result, Theorem 2. The situation is similar to that with Titchmarsh's theorem on the support of convolution [24]. If we denote by $\text{csp}(f)$ the convex hull of the spectrum of f, then for functions f and g with bounded spectra, which satisfy (7) we have

$$\text{csp}(fg) = \text{csp}(f) + \text{csp}(g) = \{x + y : x \in \text{csp}(f),\ y \in \text{csp}(g)\},$$

while in general, for arbitrary functions with bounded spectra, one can only assert that $\text{csp}(fg) \subset \text{csp}(f) + \text{csp}(g)$, and this inclusion can be proper.

We also mention that (5) and (6) is the minimal growth condition that permits to define the spectrum of a measure f in the "usual way", that is as the support of $\hat{f}$, where $\hat{f}$ is understood in the sense of distributions on the real line. An appropriate generalization of Schwarz's temperate distributions, called ω-temperate distributions, was introduced by Beurling in [6], see also [10]. Condition (5), (6) in Beurling's theory is needed to ensure the existence of test functions with bounded support. More general definition of spectrum (Definition 1) is consistent with the definition in the sense of ω-temperate distributions.

Theorem 2 can be extended to various classes of distributions, for example, to Schwartz's tempered distributions, and to distributions of Gelfand–Shilov classes S_1^β for $\beta > 2$. This is discussed in the end of Section 8.

The paper is organized as follows. In section 2 we discuss known results and conjectures about oscillation of functions with a spectral gap. The rest of the paper is formally independent of section 2. The proof of Theorem 1 occupies sections 3-7. Section 3 contains auxiliary results and Example 1. In section 4 we prove Theorem 1 under the additional assumption that f is a real analytic function and has only simple zeros on the real line. The general case is deduced in sections 5–7 by a smoothing procedure. Theorem 2 is proved in Section 8; its proof also depends on the smoothing procedure of sections 5-7. Sections 9–11 are independent of the rest of the paper. In section 9 we give a brief account of Azarin's generalization of the theory of completely regular growth, which we need for construction of Example 2 in Sections 10-11. In Section 10, a simple version of Example 2 is constructed, without the property that $|f| < \exp \omega$. Finally, in section 11, combining our method with that of Kahane and Rubel, we obtain Example 2 in its final form.

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2 History and related results

High-pass signals are important in Electrical Engineering. Statement 1 was conjectured by Logan in [27] where he proved (2) under the additional assumption that f has bounded spectrum and is bounded on the real line. One can replace in his result the condition of boundedness on the real line by the weaker condition (7). So we have the following special case of our Theorems 1 and 2.

Proposition 1 *Let $f \neq 0$ be a real function with bounded spectrum having a spectral gap $(-a, a)$. Then* (12) *holds. If in addition f satisfies* (7)*, then* (2) *holds.*

Example 2 shows that (7) is indeed needed to obtain (2). We include a simple proof of Proposition 1 based on Logan's idea.

Proof. Let b be the exponential type (bandwidth) of f, $b \geq a$. As f is real, it can be written as a sum

$$f(x) = h(x) + \overline{h}(x), \quad \text{where} \quad \overline{h}(z) := \overline{h(\overline{z})}, \tag{13}$$

and h is a function with a spectrum on $[a, b]$. According to Proposition 2 in the next section, if f satisfies (7), then h also does. Now

$$f = e^{ibx} h_1 + e^{-ibx} \overline{h_1} = \cos(bx)(h_1 + \overline{h_1}) + i \sin(bx)(h_1 - \overline{h_1}), \tag{14}$$

where h_1 and $\overline{h_1}$ have their spectra on $[a - b, 0]$ and $[0, b - a]$, respectively. We conclude that $g = h_1 + \overline{h_1}$ is a real efet with spectrum on the interval

$[(a-b),(b-a)]$, and g satisfies (7) if f does. Let $n(r,g)$ be the number of zeros of g in the disc $\{z:|z|\leq r\}$. Then Jensen's formula implies

$$\int_0^r \frac{n(t,g)}{t}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(re^{i\theta})|d\theta + O(1) \leq \frac{2(b-a)}{\pi}(r+o(r)), \quad r\to\infty. \tag{15}$$

On the other hand, (14) implies

$$f(n\pi/b) = (-1)^n g(n\pi/b),$$

from which it is easy to derive that

$$s(r,f) \geq [br/\pi] - s(r,g). \tag{16}$$

Together with (15) this implies the first inequality in (12). The second inequality follows similarly from Carleman's formula [24]. This proves the first part of the proposition.

To prove the second part we recall that according to the theorem of Cartwright and Levinson [24, Ch. V, Thm. 7], condition (7) implies that g is of completely regular growth in the sense of Levin and Pfluger. This means that the sequence of complex zeros of g in any open angle containing the positive ray has density $(b-a)/\pi$. So the upper density of positive zeros of g is at most $(b-a)/\pi$. Then (16) implies (2). □

It is important for these arguments that the spectrum of f is bounded. Our Theorem 1, whose proof is based on different ideas, extends Logan's result to functions with unbounded spectrum.

The following conjecture of P.G. Grinevich is contained in Arnold's collection [1, (1996-5)]: *"If a real Fourier integral f has a spectral gap $(-a,a)$ then the limit average density of zeros of f is at least a/π".*

In the commentary to this problem in [1], S.B. Kuksin mentioned the following result as a supporting evidence for Grinevich's conjecture. Let $\xi(t)$ be a Gaussian stationary random process, normalized by $\mathbf{E}\xi(0)=0$ and $\mathbf{E}\xi(0)^2=1$, where $\mathbf{E}$ stands for the expectation. Let r be the correlation function of this process, $r(t)=\mathbf{E}\xi(0)\xi(t)$. Assume that the function r is integrable and has a spectral gap $(-a,a)$. This implies that almost surely $\hat{\xi}(t)=0$, $t\in(-a,a)$. Denote by $\mathcal{E}_T$ the random variable which is equal to the number of zeros of the random function $\xi(t)$ on $[0,T]$. Then almost surely $T^{-1}\mathcal{E}_T$ has a limit as $T\to\infty$, and this limit is at least a/π.

Another interesting class of real functions for which the limit in (2) exists consists of the trigonometric sums

$$\sum_{k=0}^{n} a_k \cos \lambda_k x + b_k \sin \lambda_k x, \quad \text{where} \quad a_k,\ b_k\ \lambda_k \in \mathbf{R}.$$

The existence of the limit (2) for such functions can be obtained from Weyl's proof of the Mean Motion theorem [39].

Other known results deal with averaged densities (10) and (11). The earliest results on the oscillation of Fourier integrals with a spectral gap were obtained by M.G. Krein and B.Ya. Levin in the 1940-s. The following theorem is contained in [24, Appendix II, Thm 5]. *Let $f \neq 0$ be a measure of finite variation having a spectral gap $(-a, a)$. Then*

$$\liminf_{r\to\infty} \left\{ S(r, f) - \frac{2a}{\pi} r \right\} > -\infty. \tag{17}$$

This property neither follows from nor implies (2).

In a footnote on p. 403 of [24] Levin wrote: "A similar, somewhat stronger result was obtained by M.G. Krein in the theory of continuation of Hermitian-positive functions". Unfortunately, we were unable to find out what the precise formulation of Krein's result was.

Recently Ostrovskii and Ulanovskii [31] extended and improved Levin's result as follows: *Let f be a measure satisfying*

$$\int \frac{|f(x)|dx}{1 + x^2} < \infty, \tag{18}$$

and f has a spectral gap $(-a, a)$. Then

$$\liminf_{r\to\infty} \left(C(r, f) - \frac{a}{\pi} \log r \right) > 0,$$

and

$$\liminf_{r\to\infty} \left(S(r, f) - \frac{2a}{\pi} r + 3 \log r \right) > 0.$$

Condition (18) stronger than (5) and (6), but weaker than the requirement of finite variation in Levin's theorem.

These authors [30] also proved several interesting results where the assumption about a spectral gap $(-a, a)$ is replaced by a weaker assumption that the Fourier transform of f has an analytic continuation from the interval $(-a,a)$ to a half-neighborhood of this interval in the complex plane. However, in this result they characterize the oscillation of f in terms of the Beurling–Malliavin density of sign changes, which is a sort of *upper density* rather than lower density, see, for example, [22, vol. II].

The original results of Sturm in [40] were about eigenfunctions of second order linear differential operators L on a finite interval; the case of trigonometric polynomials corresponds to $L = d^2/dx^2$ on $[-\pi,\pi]$. In 1916, Kellogg [21] gave a rigorous proof of Sturm's claim for certain class of operators, whose inverses are defined by totally positive symmetric kernels on a finite interval $[a,b]$: *Let ϕ_k be the k-th eigenfunction. Then every linear combination*

$$\sum_{k=m}^{n} c_k\phi_k \not\equiv 0, \quad n > m$$

has at least $m-1$ and at most $n-1$ sign changes on $[a,b]$.

Our proof of Theorem 1 is based on a combination of two ideas: the first is the proof of Sturm's theorem from [34, III-184], the second is similar to Sturm's own argument [40, p. 430-433] (compare [33]). We recall the ideas of both proofs for the reader's convenience.

1. Write the trigonometric polynomial (1) as

$$f(x) = h(x) + \overline{h}(x), \quad \text{where} \quad h(x) = \sum_{n\geq m} c_n e^{inx},$$

then $h(x) = p(e^{ix})$ where p is a polynomial which has a root of multiplicity m at zero. By the Argument Principle, $p(z)$ makes at least m turns around zero as z describes the unit circle, so the curve $\{p(e^{ix}) : 0 \leq x \leq 2\pi\}$ crosses the imaginary axis at least $2m$ times. But $f(x) = 2\Re h(x)$ changes sign at each such crossing.

2. Use our trigonometric polynomial (1) as the initial condition of the Cauchy Problem for the heat equation on the unit circle. All coefficients will exponentially decrease with time, and the lowest degree term will have the slowest rate of decrease. On the other hand, as Sturm argued, the number of sign changes of a temperature does not increase with time, see [40, 33, 34] and Lemma 11 below. So the number of sign changes of the initial condition is at least that of the lowest degree term in its Fourier expansion.

In sections 3-4 we develop the first idea, and in sections 5-7 the second.

Other proofs of Sturm's theorem are given in [34], problems II-141, and VI-57.

To conclude this survey, we mention that the Fourier Integral first appears in Fourier's work on heat propagation [15], and that the study of sign changes was one of the main mathematical interests of Fourier during his whole career [13, 15].

3 Representation of functions on the real line as boundary values of harmonic functions

In this section we discuss representations similar to (13), which form a basis of all our arguments. Further results in this direction are contained in Section 8. A real function on the real line can be extended to a harmonic function in the upper half-plane, if the Poisson integral converges. Our observation here is that functions of much faster growth can be also nicely extended if they have a spectral gap. In this section we deal with functions that satisfy (5).

One of our tools will be the theorem of Beurling and Malliavin: *For every BMW ω and every $\eta > 0$ there exists an entire function g of exponential type η, such that $g \exp \omega$ is bounded on the real line.* The references are [8, 23] and [22, Vol. 2]. Such a function g will be called an η-multiplier. This is a deep result. However, if one imposes an additional condition that ω is even and increasing on the positive ray, the existence of a multiplier is much easier to prove, and this fact was already known to Paley and Wiener [32, p. 24-25].

There is a lot of freedom in choosing a multiplier, so we can ensure that g has some additional properties.

First, there always exists a positive multiplier. Indeed, we can replace g by $g(z)\overline{g(\overline{z})}$. A positive multiplier g allows to reduce the proof of Theorem 1 to its special case that f has finite variation. Let f be a measure satisfying the conditions of Theorem 1. For arbitrary $\eta \in (0, a)$ we choose a positive η-multiplier g. Then gf is a measure of finite variation, it has the same sequence of sign changes as f and a spectral gap $(-a+\eta, a-\eta)$ (see Lemma 3 below). Applying the special case of Theorem 1 to gf we obtain that the sequence of sign changes of f has lower density at least $(a-\eta)/\pi$, for every $\eta \in (0, a)$. This implies (2).

We will use this observation in sections 5-7.

Second, there always exists a multiplier all of whose zeros are real. (In fact, the multiplier constructed in the original proof of the Beurling and Malliavin theorem has this property). This we will use below in the proof of Proposition 2.

Suppose that $f \in L^1$. Then the Fourier transform of f is defined in the classical sense,

$$\hat{f}(t) = \int e^{-ixt} f(x)dx,$$

and $\hat{f}$ is a bounded function on the real line. For $0 < p < \infty$ we denote

$$\|h\|_p^* = \int \frac{|h(x)|^p}{1+x^2} dx,$$

and define the Hardy class H^p as the set of all holomorphic functions h in the upper half-plane with the property that $\|h(.+iy)\|_p^*$ is a bounded function of y for $y > 0$.

Lemma 1 *Let f be a real function in L^1. Then there exists a function h in $H^{1/2}$ such that*

$$f(x) = h(x) + \overline{h}(x) \quad \text{a. e.,} \quad \text{and} \quad h(iy) \to 0, \quad y \to +\infty, \tag{19}$$

where $h(x)$ is the angular limit of h. Furthermore,

$$\|h\|_{1/2}^* \leq C_1 \|f\|_1 + C_2, \tag{20}$$

where C_1 and C_2 are absolute constants. If, in addition, f is an efet, then h is an efet of Cartwright's class.

Proof. We define

$$h(z) = \frac{1}{2\pi} \int_0^\infty e^{itz} \hat{f}(t)dt, \quad \operatorname{Im} z > 0, \tag{21}$$

which is evidently holomorphic in the upper half-plane. Now we have for $\operatorname{Im} z > 0$:

$$\begin{aligned} h(z) &= \frac{1}{2\pi} \int_0^\infty e^{itz} \left\{ \int e^{-its} f(s)ds \right\} dt \\ &= \frac{1}{2\pi} \int f(s) \left\{ \int_0^\infty e^{it(z-s)} dt \right\} ds \\ &= \frac{i}{2\pi} \int f(s) \frac{ds}{z-s}. \end{aligned}$$

Taking the real part, we obtain

$$2\Re h(x+iy)=\frac{y}{\pi}\int\frac{f(s)ds}{(x-s)^2+y^2}\ ,$$

so $2\Re h$ is the Poisson integral of f. By the Cauchy–Schwarz Inequality

$$\|\Re h\|_{1/2}^{*}\leq\sqrt{\pi}\|f\|_1^{1/2}. \tag{22}$$

To prove that $h\in H^{1/2}$, we use the representation of $\operatorname{Im}h$ as a Hilbert transform,

$$\operatorname{Im}h(x+iy)=\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\epsilon}\frac{\Re h(t+iy)}{x-t}dt \tag{23}$$

and Kolmogorov's inequality,

$$m(\lambda):=\int_{|\operatorname{Im}h(x+iy)|>\lambda}\frac{dx}{1+x^2}\leq\frac{4}{\lambda}\int\frac{|\Re h(x+iy)|}{1+x^2}dx,$$

for each $\lambda>0$. These can be found in [22, v 1, p. 63]. We have

$$\begin{aligned}\|\operatorname{Im}h(.+iy)\|_{1/2}^{*}&=\int\frac{\sqrt{|\operatorname{Im}h(x+iy)|}}{1+x^2}dx\\ &= -\int_0^\infty\lambda^{1/2}\,dm(\lambda)=\frac{1}{2}\int_0^\infty\lambda^{-1/2}m(\lambda)d\lambda\\ &\leq \pi+2\|\Re h(.+iy)\|_1^{*}\int_1^\infty\lambda^{-3/2}d\lambda\leq C_1\|f\|_1+C_2.\end{aligned}$$

Combined with (22), this implies (20).

If f is an efet then $\hat{f}$ has bounded support, and (21) shows that h is also an efet. Furthermore, as $\hat{f}$ is continuous and has bounded support, we conclude from (21) that h is bounded and thus belongs to Cartwright's class. □

Remark. One can replace $H^{1/2}$ in Lemma 1 by any H^p with $p\in(0,1)$.

Now we restate the condition that $\hat{h}(t)=0$ for $t<a$ in terms of h itself.

Lemma 2 *Let $h\in H^{1/2}$ be a function represented by the Fourier integral (21), where $\hat{f}$ is bounded, and $\hat{f}(t)=0$ for $t\in(-a,a)$. Then h satisfies*

$$h(x+iy)=o(e^{-ay})\quad y\to\infty, \tag{24}$$

uniformly with respect to x.

Proof.

$$|h(x+iy)| \le \frac{\|\hat{f}\|_\infty}{2\pi} \int_a^\infty e^{-sy} ds \le \frac{e^{-ay}}{2\pi y} \|\hat{f}\|_\infty.$$

□

We denote by N the Nevanlinna class of functions of *bounded type* in the upper half-plane. A holomorphic function h in the upper half-plane belongs to N if h is a ratio of bounded holomorphic functions in the upper half-plane. We refer to [29, 35] for the theory of the class N. Function h from Lemma 1 belongs to N because $H^p \subset N$ for all $p > 0$. So we have the Nevanlinna representation

$$h(z) = e^{ia'z} B(z) e^{u(z)+iv(z)}, \tag{25}$$

where a' is a real number, B a Blaschke product, u the Poisson integral of $\log |h(x)|$, and v the conjugate function to u. In particular,

$$J(u) := \int_{-\infty}^{\infty} \frac{|u(x)|}{1+x^2} dx < \infty. \tag{26}$$

It is well-known that (25) implies

$$\limsup_{y \to +\infty} y^{-1} \log |h(iy)| = -a',$$

so (24) gives $a' \ge a$.

To generalize Lemma 1 to all functions satisfying (5) we first recall the following fact:

Lemma 3 *Let f be a measure which satisfies* (3)*, and $g \in L^\infty(\mathbf{R})$. If f has a spectral gap $(-a, a)$, and g is a function with spectrum on $[-\eta, \eta]$, $\eta < a$, then fg has a spectral gap $(-a+\eta, a-\eta)$.*

For functions f with bounded spectrum this follows from a theorem of Hurwitz, [9, Thm. 1.5.1]. In the general case, the proof is the same; we include it for the reader's convenience.

Proof. Let F and G be the generalized Fourier transforms of f and g. Then F is analytic in

$$\mathbf{C} \setminus ((-\infty, -a] \cup [a, \infty)),$$

and G is analytic in $\overline{\mathbf{C}}\backslash[-\eta,\eta]$. Let γ be a simple closed curve going once counterclockwise around the segment $[-\eta,\eta]$, then

$$g(x)=-\frac{1}{2\pi}\int_\gamma G(\zeta)e^{i\zeta x}d\zeta,$$

as in (9). For $\operatorname{Im} z<0$, we have

$$\begin{aligned}-\int_0^\infty f(x)g(x)e^{-izx}dx &= \frac{1}{2\pi}\int_0^\infty f(x)\left(\int_\gamma G(\zeta)e^{i\zeta x}d\zeta\right)e^{-izx}dx\\ =\frac{1}{2\pi}\int_\gamma G(\zeta)\int_0^\infty f(x)e^{i(\zeta-z)x}dx\,d\zeta &= \frac{1}{2\pi}\int_\gamma G(\zeta)F(z-\zeta)d\zeta. \qquad (27)\end{aligned}$$

This function is analytic in

$$\mathbf{C}\backslash\left((-\infty,-a+\eta]\cup[a-\eta,\infty)\right).$$

A similar computation for

$$\int_{-\infty}^0 f(x)g(x)e^{-izx}dx$$

gives the same result (27). □

We state our conclusions as

Proposition 2 *Let f be a locally integrable function satisfying the conditions of Theorem 1. Then*

$$f=h+\overline{h}\quad a.\ e.,$$

where h is a function of bounded type in the upper half-plane, having representation (25) in which $a'\geq a$. If f is an efet then h can be chosen in Cartwright's class.

Proof. Choose $\eta\in(0,a)$. Let g be an η-multiplier that is real on the real line and has only real zeros. Then g belongs to Cartwright's class, and thus has completely regular growth, which implies

$$\log|g(re^{i\theta})|=\eta r\sin\theta+o(r)\quad r\to\infty, \qquad (28)$$

uniformly with respect to θ for $|\theta| \in (\epsilon, \pi - \epsilon)$, for every $\epsilon > 0$. Furthermore, $gf \in L^1$ by (5), and gf has a spectral gap $(-a + \eta, a - \eta)$ by Lemma 3. According to Lemma 1,

$$gf(x) = h_1(x) + \overline{h_1}(x), \tag{29}$$

where $h_1 \in H^{1/2}$, so $h_1 \in N$. Inequality (24) implies that

$$\log |h_1(re^{i\theta})| \leq (\eta - a) r \sin\theta + o(r) \quad r \to \infty, \tag{30}$$

uniformly with respect to θ. Dividing (29) by g (which has no zeros outside the real axis), we conclude that (19) holds with $h = h_1/g$ which evidently belongs to N. Now (28) and (30) show that

$$\log |h(re^{i\theta})| \leq -ar \sin\theta + o(r) \quad r \to \infty,$$

uniformly with respect to θ, which implies that $a' \geq a$ in (25). This proves the first statement.

If f is an efet, let b be its exponential type, and F its generalized Fourier transform. Then F is analytic in

$$\overline{\mathbf{C}} \backslash ([-b, -a] \cup [a, b])$$

and $F(\infty) = 0$. By the theorem on separation of singularities,

$$F = F_1 + F_2, \tag{31}$$

where F_1 is analytic in $\overline{\mathbf{C}} \backslash [-b, -a]$, F_2 is analytic in $\overline{\mathbf{C}} \backslash [a, b]$, and $F_j(\infty) = 0,\ j = 1, 2$. Such decomposition of F is unique. Taking the inverse transforms as in (9), we obtain

$$f(x) = h^+(x) + h^-(x), \tag{32}$$

where $h^{\pm}$ are efet with spectra on $[-b, -a]$ and $[a, b]$, respectively, so $h^{\pm}(iy) = O(\exp(-a|y|)),\ y \to \pm\infty$. Multiplying (32) by g we obtain

$$gf(x) = gh^+(x) + gh^-(x),$$

where $gh^{\pm}$ are efet with spectra on positive and negative rays, respectively. On the other hand, according to Lemma 1, (29) is another representation of gf as a sum of two efet with spectra on positive and negative rays. As such representation is unique, because the decomposition (31) is unique and the correspondence $f \mapsto F$ is one-to-one, we obtain $gh^+ = h_1$. So h^+ belongs to Cartwright's class, as an entire function that is a ratio of two entire functions of Cartwright's class. This follows from a theorem of Krein [25, Ch. 16, Thm. 1] which says that an entire function belongs to Cartwright's class if and only if its restrictions to the upper and lower half-planes belong to the class N.

□

Construction of Example 1. We combine Logan's method [27, Thm 5.5.1] with the theorem of Beurling and Malliavin. Without loss of generality, we may assume that $a = \pi - 2\epsilon$, and $b = \pi + 2\epsilon$, where $\epsilon > 0$. Let g_1 be a real entire function of zero exponential type, satisfying (7), with only simple zeros, and such that the zero set of g_1 coincides with the set of integer points on the intervals $[y_k, x_k]$:

$$g_1(n) = 0,\ g'(n) \neq 0 \quad \text{for} \quad n \in \mathbf{Z} \cap \left(\cup_{k=1}^{\infty} [y_k, x_k]\right).$$

Such function g_1 can be easily constructed if the intervals $[y_k, x_k]$ are not too long in comparison with x_k, for example, if

$$\sum_{k=1}^{n} (x_k - y_k) \leq x_n^{\alpha} \quad \text{for some} \quad \alpha \in (0, 1).$$

One can obtain longer intervals, whose size can be characterized in terms of the Beurling–Malliavin density [22, vol. II]. Let g be an entire function of exponential type ϵ, which is positive on the real line and such that $|x|^2 g(x) g_1(x)$ is bounded for $x \in \mathbf{R}$. Such a function g exists because g_1 satisfies (7). Then

$$f_1(z) = g(z) g_1(z) \sin \pi z$$

does not change sign on any of the intervals $[y_k, x_k]$, and $\hat{f}_1$ has support on

$$[-\pi - \epsilon, -\pi + \epsilon] \cup [\pi - \epsilon, \pi + \epsilon].$$

Evidently, $f_1 \in L^1$. To destroy the multiple zeros of f_1 on the intervals $[y_k, x_k]$, we put $f(z) = f_1(z + 1/2) + f_1(z)$. □

4 Theorem 1 for real analytic functions

In this section we prove Theorem 1 for real analytic functions f whose real zeros are simple, so that the sign changes occur exactly at the zeros of f.

The general case will be obtained from this special one in sections 5–7, by a smoothing procedure.

Proof of Theorem 1 for real analytic functions. We write, as in Proposition 2,

$$f(x) = h(x) + \overline{h}(x), \tag{33}$$

where h has spectrum on $[a, \infty)$, and consider the Nevanlinna representation (25). We define $v(x)$ for real x as the limit from the upper half-plane. Our assumptions about analyticity and simple zeros imply that v in (25) is piecewise continuous; the only jumps of $-\pi$ occur exactly at the real zeros of h (which are all simple).

Put

$$\phi(x) = \arg h(x) := a'x + \arg B(x) + v(x).$$

The Blaschke product

$$B(z) = \prod_{n \geq 1} \left(1 - \frac{z}{z_n}\right)\left(1 - \frac{z}{\overline{z}_n}\right)^{-1}, \tag{34}$$

has a continuous argument because zeros in the upper half-plane cannot accumulate to points on the real axis. Furthermore, $\arg B$ is an increasing function, which is seen by inspection of each factor of the product (34).

Let γ be the curve in the (x, y)-plane consisting of the graph of ϕ and vertical segments of length π added at the points of discontinuity of v. At each intersection of this curve with the set

$$L = \{(x, y) : x \in \mathbf{R},\, y - \pi/2 \in \pi\mathbf{Z}\}, \tag{35}$$

the number $h(x)$ is purely imaginary, that is $f(x) = 0$ by (33).

So we want to estimate from below the number of intersections of γ with L over the intervals $(0, r)$.

We fix $\epsilon \in (0, 1/2)$ and prove that on every interval $[(1-\epsilon)x, x]$ with x large enough there exists a point x' such that

$$\phi(x') \geq a'x' + v(x') > (a' - 2\epsilon)x'. \tag{36}$$

It immediately follows from (36) that the lower density of the number of intersections $\gamma \cap L$ is at least a'/π. So it remains to prove (36).

We recall that v is harmonically conjugate to u, and that u satisfies (26). According to Kolmogorov's inequality [22, v 1, p. 63]

$$\int_{|v(x)|>\lambda} \frac{dx}{1+x^2} \leq \frac{4}{\lambda} \int_{-\infty}^{\infty} \frac{|u(x)|}{1+x^2} dx,$$

for each $\lambda > 0$.

We break u into two parts with disjoint supports, $u = u_0 + u_1$, where the support of u_0 belongs to $[-r_0, r_0]$ for some $r_0 > 0$ and u_1 satisfies

$$\int \frac{|u_1(x)|}{1+x^2} dx = \int_{|x|>r_0} \frac{|u_1(x)|}{1+x^2} dx < \epsilon^2/8, \tag{37}$$

which is possible in view of (26). Let $v_j = \mathcal{H}u_j,\ j = 0, 1$; where $\mathcal{H}$ stands for the Hilbert transform,

$$\mathcal{H}u(x) = \lim_{y \to 0+} \frac{1}{\pi} \left(\int \frac{x-t}{(x-t)^2 + y^2} + \frac{t}{t^2+1} \right) u(t)\, dt.$$

Lemma 4 $|v_0(x)| \leq J(2r_0 + r_0^{-1})/\pi$ *for* $|x| > 2r_0$, *where* $J = J(u)$ *is defined in* (26).

Proof.

$$\begin{aligned} |v_0(x)| &\leq \frac{1}{\pi} \left| \int_{-r_0}^{r_0} \frac{u_0(t)}{x-t} dt \right| + \frac{1}{\pi} \left| \int_{-r_0}^{r_0} \frac{t u_0(t)}{t^2+1} dt \right| \\ &\leq \frac{1}{\pi} r_0^{-1} \int_{-r_0}^{r_0} |u_0(t)| dt + \frac{1}{\pi} r_0 J \\ &\leq \frac{1}{\pi} r_0^{-1} (1 + r_0^2) J + \frac{1}{\pi} r_0 J \\ &= J(2r_0 + r_0^{-1})/\pi. \end{aligned}$$

□

Now we prove that for every $x > 2$ there exists

$$x' \in [(1-\epsilon)x, x],$$

such that

$$v_1(x') > -\epsilon x. \tag{38}$$

Suppose that this is not so. Then we apply Kolmogorov's inequality to v_1 and u_1 with $\lambda = \epsilon x$ and (37):

$$\int_{(1-\epsilon)x}^{x} \frac{dt}{2t^2} < \int_{(1-\epsilon)x}^{x} \frac{dt}{1+t^2} < \frac{4}{\epsilon x} \int \frac{|u_1(x)|}{1+x^2} dx < \frac{\epsilon}{2x}.$$

Evaluating the integral on the left we conclude $\epsilon/(1-\epsilon) < \epsilon$, a contradiction. Lemma 4 and (38) imply (36). This proves (our special case of) Theorem 1.

For the future use we record a more quantitative version of the result we just proved:

Proposition 3 *Let f be a function satisfying the conditions of Theorem 1. Suppose that f is real analytic and has only simple zeros on the real line. Write $f = h + \overline{h}$ as in* (33)*, and let h be represented by the formula* (25)*, with $J = J(u)$ as in* (26)*. Suppose that*

$$\int_{|x|>r_0} \frac{|\log|h(x)||}{1+x^2} < \epsilon^2/8$$

for some $r_0 > 1$ and $\epsilon \in (0, 1/2)$. Then

$$s(r, f) \geq (a-\epsilon)r/\pi - J(2r_0 + r_0^{-1})/\pi - 1 \quad \textit{for} \quad r > 2r_0.$$

□

5 Heating

In this section we assume that f is a measure of finite variation.

If f is not real analytic, or is real analytic and has multiple zeros on the real line, we "heat" it. This means that we replace our f by the convolution[2] with the heat kernel,

$$f_t = K_t * f, \quad f_0 = f, \tag{39}$$

$$K_t(x) = \frac{1}{\sqrt{\pi t}} e^{-x^2/t}.$$

[2]In the works on the heat equation this is called a Poisson integral. We *don't do this* to avoid confusion with the harmonic Poisson integral.

Evidently, the f_t are real analytic with respect to x for all $t > 0$. All

$$\hat{f}_t = \hat{K}_t \hat{f} = \exp(-s^2 t/4)\hat{f}$$

have the same support because $\hat{K}_t$ never vanishes.

Laguerre and Pólya [33, 34] proved that the number of sign changes of f_t is at most that of f. (This assertion was stated by Sturm for the case of a finite interval. The proof given in [34] shows that the result is in fact a generalization of the “Rule of signs” of Descartes.) We cannot use this result directly because our functions have infinitely many sign changes, and we have to control their number on every interval $(0, r)$. So in the next two sections we will prove the necessary generalization of the Laguerre–Pólya's theorem.

In this section we show that for functions f_t, the conditions of Proposition 3 hold uniformly with respect to t.

Lemma 5 *Let f be a measure of finite variation with a spectral gap $(-a, a)$, and $f_t = K_t * f$. Define h_t by (21) using f_t instead of f.*

Then there exists $t_0 > 0$ such that $J(\log |h_t|) \leq C_1$ for $t \in (0, t_0)$, where C_1 is independent of t. Furthermore, for every $\epsilon > 0$ there exists $r_0 > 0$ such that for all $t \in (0, t_0)$ we have

$$\int_{|x| \geq r_0} \frac{|\log |h_t(x)||}{1 + x^2} dx < \epsilon. \tag{40}$$

We emphasize that r_0 and C_1 are independent of t. They only depend of h and ϵ.

Proof. First of all,

$$\|f_t\|_1 = \int |K_t * f|(x)dx \leq \int (K_t * |f|)(x)dx = \int |f(x)|dx,$$

so $\|f_t\|_1$ does not exceed the total variation. Using Lemma 1 we obtain $\|h_t\|^*_{1/2} \leq C$, with C independent of t. Thus

$$\sqrt{|h_t(x)|} = k_t(x)(1 + x^2), \quad \text{where} \quad \|k_t\|_1 \leq C. \tag{41}$$

We have

$$\begin{aligned} \log^+ |h_t| &\leq 2 \log^+ |k_t| + 2 \log(1 + x^2) \\ &\leq 2|k_t| + 2 \log(1 + x^2) \end{aligned} \tag{42}$$

Let $u_t(x) = \log|h_t(x)|$ for real x and $t \geq 0$. Dividing (42) by $1+x^2$, integrating and using (41) gives

$$J(u_t^+) = \int \frac{u_t^+(x)}{1+x^2}dx < C, \tag{43}$$

where C is independent of t. Similarly we obtain from (42) that

$$\int_{|x|\geq r_0} \frac{u_t^+(x)}{1+x^2}dx < \frac{2}{(1+r_0)^{1/4}} \left(\|k_t\|_1 + \int_0^\infty \frac{\log(1+x^2)}{(1+x^2)^{3/4}}dx \right) < \epsilon,$$

with some $r_0 > 1$ independent of t.

Property (43) makes it possible to extend u_t^+ to the upper half-plane by Poisson's formula. We denote the extended function by w_t. Notice that $h_t \in N$ for all t, and $w_t(x+iy) - ay$ is a positive harmonic majorant of $\log|h_t|$ in the upper half-plane.

Now we prove

$$J(u_t^-) < C, \tag{44}$$

with C independent of t, and

$$\int_{|x|\geq r_0} \frac{u_t^-(x)}{1+x^2}dx < \epsilon \tag{45}$$

for some $r_0 > 0$. Fix a point z_0 in the upper half-plane, such that $\delta = |h(z_0)| > 0$. As $h_t(z_0) \to h(z_0)$ as $t \to 0$, we conclude that $h_t(z_0) > \delta/e$ when t is small enough. Let b be the *true left end* of the support of $\hat{h}_t$. It is important to notice that b is *independent* of t, because $\hat{h}_t = \hat{K}_t\hat{h}$. Then

$$u_t(z_0) - b\mathrm{Im}\, z_0 \geq \log\delta - 1 > -\infty, \tag{46}$$

when t is small enough. Here we mean that u_t is extended to a harmonic function in the upper half-plane by the Poisson integral. Now (46) implies (44). It remains to prove (45). For psychological reasons it is better to work in the unit disc $\mathbf{U}$ instead of the upper half-plane. The fractional-linear transformation $T(z) = (z-i)/(z+i)$ maps the upper half-plane onto $\mathbf{U}$, $T(\infty) = 1$, and we put $\zeta_0 = T(z_0)$, and

$$w_t = u_t \circ T^{-1} - b\mathrm{Im}\, T^{-1}. \tag{47}$$

As a consequence of (46) we have

$$w_t(\zeta_0) \geq \log\delta - 1 > -\infty. \tag{48}$$

The measure $dx/(1+x^2)$ on the real line corresponds to the measure $d\theta$ on the unit circle $\mathbf{T} = \{e^{i\theta} : \theta \in \mathbf{R}\}$.

It follows from (43) that each w_t is a difference of positive harmonic functions in the unit disc, so it is the Poisson integral of some charge μ_t of finite variation on the unit circle. The constant b in (47) comes from the Nevanlinna representation

$$h_t(z) = e^{ibz}B_t(z)e^{u_t(z)+iv_t(z)}$$

similar to (25), and b does not depend on t. So all charges μ_t have an atom of mass exactly $-b$ at the point 1.

Let $\mu_t = \mu_t^+ - \mu_t^-$ be the Jordan decompositions. Conditions (48) and (43) imply that μ_t are of bounded total variation, with a bound independent of t. So we have weak convergence $\mu_t \to \mu_0,\ t \to 0$. Let ϕ be a positive continuous function on the unit circle, which is identically equal to 1 in some neighborhood of the point 1, and at the same time

$$\left| b - \int_{\mathbf{T}} \phi|\mu_0| \right| < \epsilon/2,$$

where $|\mu_0| = \mu_0^+ + \mu_0^-$ is the variation of μ_0. Then there exists t_0 such that

$$\left| b - \int_{\mathbf{T}} \phi|\mu_t| \right| < \epsilon, \quad \text{for} \quad 0 \leq t \leq t_0. \tag{49}$$

When translated back to the real line from the unit circle, this implies (45).
□

6 Preliminaries on temperatures

Here we collect for the reader's convenience some facts about convolutions (39) of distributions with the heat kernel. We assume that f is a linear functional on some space of infinitely differentiable test functions which contains all functions $\phi_{t,y,k}(x) = K_t^{(k)}(x-y)$, for example, f can be a Schwartz's tempered distribution. We use the convenient notation[3]

$$u(x,t) = f_t(x) \tag{50}$$

and consider u in the upper half-plane $\{(x,t) : t \geq 0, x \in \mathbf{R}\}$.

The function u in (50) is a solution of the heat equation in the open upper half-plane:

$$4\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{51}$$

Such functions are called *temperatures.* Formula (39) solves the initial value problem on an infinite rod (the x-axis) with given initial temperature $f(x)$. A standard reference on the subject is [12]. Here is the precise statement about the boundary behavior of u which is a slight generalization of [12, 1.XVI.7]:

Lemma 6 *Let f be a positive measure, and $u(x,t) = (K_t * f)(x)$. Then for every $x \in \mathbf{R}$,*

$$\liminf_{t\to 0} u(x,t) \geq \liminf_{\epsilon \to 0+} \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(t)dt.$$

This is a general property of convolutions with positive symmetric kernels. It follows for a locally integrable f that at every Lebesgue density point x of f, the limit $\lim_{t\to 0+} u(x,t)$ exists and equals $f(x)$.

Next we consider temperatures whose initial data are derivatives of delta functions:

$$u_{n,a}(x,t) = (K_t * \delta^{(n)})(x-a) = \frac{1}{\sqrt{\pi t}}\frac{d^n}{dx^n}e^{-(x-a)^2/t}, \quad a \in \mathbf{R}. \tag{52}$$

Lemma 7 *For every integer $n \geq 0$ there exist $t_n > 0$ and two curves $\gamma_n^+(a)$ and $\gamma_n^-(a)$, which are graphs of functions*

$$x = a + g_n^{\pm}(t),\ 0 \leq t \leq t_n,\ g_n^{\pm}(0) = 0,\ g_n^+ \geq 0,\ g_n^- \leq 0,$$

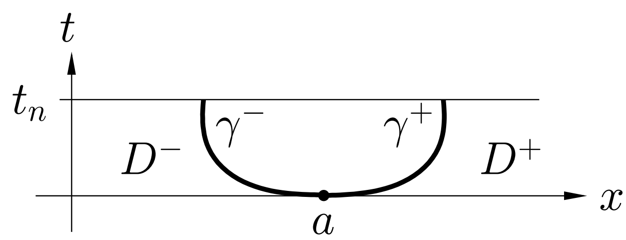
with the following properties: $|u_{n,a}(x,t)| = 1$, $(x,t) \in \gamma_n^+(a) \cup \gamma_n^-(a)$ and $|u_{n,a}(x,t)| < 1$ in the two regions

$$D^-(a) = \{(x,t) : x < a + g_n^-(t),\ 0 < t < t_n\}$$

[3] We apologize for such abuse of the letter u, but the harmonic function u of sections 3-5 will not appear anymore until the end of section 7.

and

$$D^+(a) = \{(x,t) : x > a + g_n^+(t),\ 0 < t < t_n\}.$$



Proof. We assume that $a = 0$. Then

$$u_n(x,t) = \frac{(-1)^n}{\sqrt{\pi}} t^{-(n+1)/2} e^{-x^2/t} H_n(x/\sqrt{t}),$$

where H_n are the Hermite polynomials. It is clear from this formula that for $t_n > 0$ small enough, and for every $t \in (0, t_n)$, the equation $|u_n(x,t)| = 1$ has $2n+2$ roots $x_1(t) < x_2(t) < \ldots < x_{2n+2}(t)$, and that the curves $\gamma_n^+(0) = \{(x_{2n+2}(t), t) : t \in (0, t_n)\}$ and $\gamma_n^-(0) = \{(x_1(t), t) : t \in (0, t_n)\}$ have all the required properties.

□

The next lemma (due to L. Nirenberg) is called the Strong Minimum Principle [12, 1.XV.5]

Lemma 8 *Let D be a bounded region in the horizontal strip $P = \{(x,t) : 0 < t < T\}$, and u a temperature in D. Suppose that*

$$\liminf_{s \to \sigma} u(s) \geq 0, \quad \textit{for all} \quad \sigma \in \partial D \cap (P \cup (\mathbf{R} \times \{0\})). \tag{53}$$

Then $u \geq 0$ in D, and if $u(s) = 0$ for some point $s \in D$ then $u \equiv 0$ in D.

We need an extension of the Minimum Principle, analogous to the Phragmén–Lindelöf Theorem in the theory of harmonic functions:

Lemma 9 *Let D be a region as in Lemma 8, and u a temperature in D. Suppose that u is bounded from below, and that* (53) *holds for all but finitely many points $\sigma \in \partial D \cap (\mathbf{R} \times \{0\})$. Then the same conclusion as in Lemma 8 holds.*

Proof. Let $x_1, x_2, \ldots, x_n$ be the exceptional points on the real axis. Consider the auxiliary function

$$w(s) = \begin{cases} \sum_{k=1}^{n} \log^+ \dfrac{1}{|s - x_k|}, & s \in \mathbf{R} \times \{0\}, \\ \\ (K_t * w(.,0))(x), & s = (t,x),\ x \in \mathbf{R}, t > 0. \end{cases}$$

Then w is a positive temperature in P, and

$$w(s) \to +\infty \quad \text{as} \quad s \to x_k,\ \ s \in P,\ \ 1 \le k \le n.$$

So, for every $\epsilon > 0$, the function

$$u_\epsilon = u + \epsilon w$$

satisfies all conditions of Lemma 8. Thus $u_\epsilon \ge 0$, that is $u(z) \ge -\epsilon w(z)$. Letting $\epsilon \to 0$, we conclude that $u \ge 0$. So u satisfies the conditions of Lemma 8, and the conclusions of Lemma 8 hold for u. □

Lemma 10 *Let u be a temperature in some region D of the (x,t)-plane. Then multiple zeros of the functions $x \mapsto u(x,t)$ are isolated in D.*

Proof. Suppose that u has a non-isolated multiple zero. Let $m \ge 2$ be the minimum of the multiplicities of such zeros. Then there exists an analytic germ $g(t)$ which gives the position of such a multiple zero for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some t_0 and $\epsilon > 0$. So we have

$$u(x,t) = (x - g(t))^m v(x,t),$$

in a neighborhood of $(g(t_0), t_0)$. Here v is a real analytic function

$$v(g(t_0), t_0) \ne 0.$$

We differentiate, and see that the lowest order term in $\partial^2 u / \partial x^2$ is

$$m(m-1)(x - g(t))^{m-2} v(x,t),$$

while all terms in $\partial u / \partial t$ are of order at least $m - 1$. So u cannot satisfy the heat equation. □

7 Sign changes of distributions and proof of Theorem 1

We recall that the *number of sign changes* $s((a,b),f)$ of a real distribution f on an open interval (a,b) is defined as the infimum of degrees of real polynomials $p \neq 0$ such that the restriction of pf on (a,b) is positive. If no such polynomials exist we set $s((a,b),f) = \infty$. This definition applies to those distributions which can be restricted to intervals. So we assume that the space of test functions contains functions with bounded support, and that such test functions are dense in $C_0^\infty(I)$ for every interval I.

It follows that every positive distribution is a measure on (a,b). We conclude that distributions with finitely many changes of sign on an interval have finite order on this interval.

(A) *In what follows we consider only distributions that have finitely many changes of sign on every interval.*

Let p be a polynomial such that $\deg p = s((a,b),f)$, and $pf \geq 0$ on (a,b), and let $x_1 < x_2 < \ldots < x_n$ be all roots of p (disregarding multiplicities). Then it is clear that the restriction of f on every component of the set $(a,b)\backslash\{x_k,\ldots,x_n\}$ is a (positive or negative) measure. The total measure of a component may be infinite.

The following result is classical:

Lemma 11 *Let f be a distribution with bounded support. Then for every $t > 0$, the function $K_t * f$ has no more changes of sign on $(-\infty,\infty)$ than f has.*

Proof. We fix $t > 0$ and write

$$f_t(y) = (K_t * f)(y) = \frac{1}{\sqrt{\pi t}}\left(f(x), e^{-(x-y)^2/t}\right),$$

or

$$\sqrt{\pi t}\, e^{y^2/t} f_t(y) = \left(e^{-x^2/t} f(x), e^{2xy/t}\right).$$

Multiplication by an exponential does not change the number of sign changes, so the problem is reduced to proving that

$$g(u) = (G(v), e^{uv})$$

has at most as many sign changes as G. Here we made the change of variable $v = x\sqrt{2/t},\ u = y\sqrt{2/t}$.

Now we repeat the argument which Pólya and Szegö credit to Laguerre, [34, Problem V-80]. We argue by induction on the number of sign changes of G. If G has a constant sign then evidently g does. Now suppose that for every G with at most n changes of sign, g has at most n changes of sign. If G has $n+1$ changes of sign, then there exists α such that $G(v)(v-\alpha)$ has at most n changes of sign. Then the real analytic function

$$(G(v)(v-\alpha), e^{uv}) = g'(u) - \alpha g(u).$$

has at most n changes of sign by the induction assumption. But the right hand side is

$$e^{\alpha u}\frac{d}{du}\left(g(u)e^{-\alpha u}\right),$$

and we conclude from Rolle's theorem that g has at most $n+1$ changes of sign. □

Lemma 12 *Let Q be an open quadrilateral in the (x,t)-plane, whose boundary consists of two horizontal intervals: $[a,b]$ on the x-axis and $[c,d]$ on the line $t = T$, and two simple curves, γ^+ connecting a to c and γ^- connecting b to d. Let u be a temperature in Q, which is bounded from above or from below in Q and real analytic on $\overline{Q}\backslash\{a,b\}$. Assume that $u(x,t) \neq 0$ for $(x,t) \in \gamma^+ \cup \gamma^-$. Then the number of sign changes of $u(.,T)$ on (c,d) is less than or equal to the number of sign changes of $u(.,0)$ on (a,b) plus 4.*

Proof. Consider a maximal open interval $\ell \subset (c,d)$ of sign constancy of $u(.,T)$. Let $\varepsilon \in \{1,-1\}$ be the sign of u on ℓ. Let D be the connected component of the set

$$\{(x,t) \in Q : \varepsilon u(x,t) > 0\} \quad \text{such that} \quad \ell \subset \partial D.$$

Notice that $u(x,t) = 0$ for $(x,t) \in \partial D \cap Q$. We claim that

$$\partial D \cap [c,d] = \overline{\ell}. \tag{54}$$

Indeed, on those maximal open intervals of $[c,d]$ of sign constancy, which are adjacent to ℓ, the sign is $-\varepsilon$, so these intervals cannot intersect ∂D. Suppose that there is an interval of constant sign ε, say $\ell^* \subset [c,d]$, such that $\ell^* \subset \partial D$, and $\ell^* \cap \ell = \emptyset$. Then there is a maximal interval ℓ' of constant sign $-\varepsilon$ between ℓ and ℓ^*.

Then the component D' of the set $\{s \in Q : \varepsilon u(s) < 0\}$ whose boundary contains ℓ' has its closure in the upper half-plane (being separated by D from the x-axis), and this contradicts Lemma 8. This proves our claim (54).

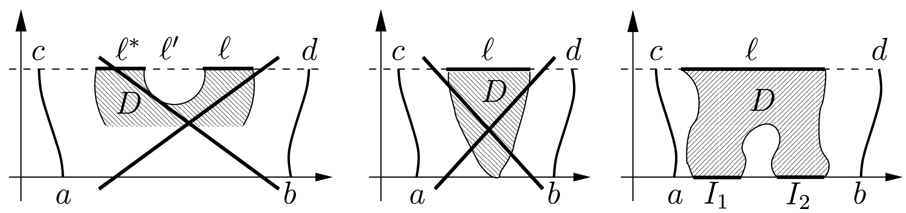


Figure 1: First two possibilities are excluded by the Minimum Principle

If $\partial D \backslash \ell \subset Q$, then $u \equiv 0$ in D by Lemma 8, and thus $u \equiv 0$ in Q, contradicting our assumptions. It is clear that ∂D can intersect a lateral side γ^+ or γ^- of Q at a point (x, t) with $t > 0$ only if ∂D contains this lateral side; this happens for exactly two domains D which we call *lateral* domains.

Suppose that the boundary of a component D contains an open interval of the x-axis. Then it contains a maximal open interval $I \subset (a, b)$ of sign constancy of $u(., 0)$. As $u(., 0)$ is real analytic on (a, b), it is easy to see that the boundaries of two different components D_1 and D_2 cannot contain the same interval I.

Now we consider non-lateral components D whose boundaries do not contain any interval on the x-axis. We claim that the number of such components is at most 2. For every such component D, the intersection of ∂D with the interval (a, b) of the x-axis is a discrete set. It follows from Lemma 8 that the intersection of ∂D with the x-axis contains a or b. Suppose without loss of generality that u is bounded from above. Let D be a non-lateral domain whose boundary contains a. Then Lemma 9 implies that u is negative in D. There can be at most one such component D. Indeed, if there are two, say D' and D'' there would be a component D^* between D' and D'', such that u is positive in D^*, and such that the boundary of D^* intersects the real axis at the point a alone. This proves our claim.

So every component D, except two lateral ones and possibly at most two other components whose boundaries contain a and b, has a maximal interval

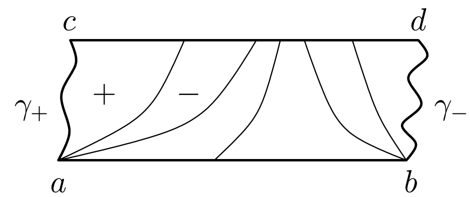


Figure 2: Partition of Q into components D.

I of sign constancy of $u(.,0)$ on its boundary. It follows that the total number of domains D is at most the number of intervals I plus 4. This proves the lemma. □

Remark. It is enough to require in this lemma that u is bounded from one side near A and from another side near B. We will use this remark in the proof of Theorem 2′ in Section 8.

The points of sign changes cannot be uniquely defined, even for continuous functions, because a function can be zero on an interval. To remove this arbitrariness, we set, as before,

$$s(r,f) = \begin{cases} s((0,r),f) & \text{for} \quad r > 0, \\ \\ s((r,0),f) & \text{for} \quad r < 0. \end{cases}$$

Then $s((-r,r),f) = s(r,f) + s(-r,f) + \text{const}$, with the constant depending of f. Now $s(r,f)$ is an integer-valued function, increasing for $r > 0$ and decreasing for $r < 0$, and according to our assumption (A) it is everywhere finite. So the points of jump of $s(r,f)$ are isolated, and the number of sign changes is zero on every interval of constancy of $s(r,f)$. If f is a measure then the magnitude of jumps of $s(r,f)$ is at most two.

Let $d\mu$ be a measure on an interval containing a point x_0. We say that x_0 is a *concentration point* of $d\mu$ if

$$\liminf_{\delta \to 0} \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} |d\mu| \in (0, \infty].$$

A standard argument with Besicovich's covering lemma [16] shows that concentration points exist for every non-zero measure.

Proposition 4 *Let f be a distribution with finitely many changes of sign on every finite interval, and such that the convolution with the heat kernel is defined. Let $a < b$ be two jump points of $s(r, f)$ and $\epsilon \in (0, (b-a)/2)$. Then there exist α and β in ϵ-neighborhoods of a and b, respectively, and $t_0 > 0$ such that*

$$s((\alpha, \beta), K_t * f) \leq s((\alpha, \beta), f) + 4, \quad \textit{for} \quad t \in (0, t_0). \tag{55}$$

Proof. Jump points of $s(r, f)$ are isolated, so we may assume from the beginning that a and b are the only jump points in their ϵ-neighborhoods $U_a(\epsilon)$ and $U_b(\epsilon)$. We put $U_a^* = U_a(\epsilon/2) \backslash \{a\}$ and $U_b^* = U_b(\epsilon/2) \backslash \{b\}$. The restriction of f on each component of $U_a^* \cup U_b^*$ is a positive or negative measure. Now we consider four cases, depending on whether the restrictions of f on U_a^* and U_b^* are zero measures or non-zero measures.

Case 1. Restrictions of f on both U_a^* and U_b^* are non-zero measures. We choose $\alpha \in U_a^*$ and $\beta \in U_b^*$ to be concentration points of these restrictions. Consider a smooth function η, $0 \leq \eta \leq 1$ with the following properties: $\eta(x) = 1$ for x in a neighborhood of $[\alpha, \beta]$, and $\operatorname{supp} \eta \subset (\alpha_1, \beta_1)$ where $\alpha_1 < \alpha$ and $\beta_1 > \beta$ are chosen to satisfy the condition that

$$s((\alpha, \beta), f) = s((\alpha_1, \beta_1), f),$$

so that

$$s((\alpha, \beta), f) = s((-\infty, \infty), \eta f). \tag{56}$$

Then, according to Lemma 6, there exists $t_0 > 0$ and $\delta > 0$, such that

$$|K_t * (\eta f)(\beta)| \geq \delta > 0 \quad \text{for} \quad t \in (0, t_0) \tag{57}$$

and (as $1 - \eta(x) = 0$ in a neighborhood of β)

$$|K_t * ((1 - \eta) f)(\beta)| \leq \delta/2, \quad \text{for} \quad t \in (0, t_0), \tag{58}$$

and similar inequalities hold with α instead of β.

Consider the functions

$$\begin{aligned} g_{t,\tau} &= K_t * (K_\tau * (\eta f) + (1 - \eta) f) && (59) \\ &= K_{t+\tau} * (\eta f) + K_t * ((1 - \eta) f), && (60) \end{aligned}$$

$$\text{where} \qquad 0 \leq t \leq t_0/2 \quad \text{and} \quad 0 \leq \tau \leq t_0/2.$$

We recall that by definition $K_0 * f = f$. Using (56) and Lemma 11, we obtain for $\tau \in (0, t_0/2)$:

$$\begin{aligned} s((\alpha,\beta),f) &= s((-\infty,\infty),\eta f) \geq s((-\infty,\infty),K_\tau * (\eta f)) \\ &\geq s((\alpha,\beta),K_\tau * (\eta f)) = s((\alpha,\beta),g_{0,\tau}). \end{aligned} \tag{61}$$

Now, in view of (57), (58) and similar inequalities at the point α, the function

$$u_\tau(x,t) = g_{t,\tau}(x) \tag{62}$$

satisfies

$$|u_\tau(\alpha,t)| \geq \delta/2 \quad \text{and} \quad |u_\tau(\beta,t)| \geq \delta/2$$

for all $\tau \in (0, t_0/2)$ and all $t \in (0, t_0/2)$. So Lemma 12 is applicable to the rectangle $(\alpha, \beta) \times (0, t_0/2)$ and to the function $u_\tau(x,t)$, and we obtain

$$s((\alpha,\beta),g_{t,\tau}) \leq s((\alpha,\beta),g_{0,\tau}) + 4, \quad \text{for} \quad \tau \in (0,t_0/2)$$

Using (61), passing to the limit when $\tau \to 0$ and taking into the account that the number of sign changes does not increase in the limit, we obtain

$$s((\alpha,\beta),K_t * f) = s((\alpha,\beta),g_{t,0}) \leq s((\alpha,\beta),f) + 4.$$

Case 2. Now suppose that the restriction of f on U_a^* is zero, while the restriction on U_b^* is a non-zero measure. Then we choose $\beta \in U_b$ as in the Case 1. To choose α, we recall that the restriction of f on U_a coincides with $\delta_a^{(n)}$ for some integer $n \geq 0$. Let η, $0 \leq \eta \leq 1$ be an infinitely differentiable function with the properties that $\eta(x) = 1$ for x in a neighborhood of $[a + \epsilon/2, \beta]$, and $\eta(x) = 0$ for x outside the interval $[a + \epsilon/3, \beta_1]$, $\beta_1 > \beta$, so that

$$s((a,\beta),f) = s((-\infty,\infty),\eta f). \tag{63}$$

Consider the curve $\gamma_n^+(a)$ defined in Lemma 7. It follows from Lemmas 6 and 7 that there exist $t_0 > 0$ and $\delta > 0$ such that

$$|K_t * ((1-\eta)f)(x)| \geq \delta \quad \text{for} \quad t \in (0,t_0),\ (x,t) \in \gamma_n^+(a) \tag{64}$$

and

$$|K_t * (\eta f)(x)| \leq \delta/2 \quad \text{for} \quad t \in (0,t_0),\ (x,t) \in \gamma_n^+(a), \tag{65}$$

while on the end β we have (57) and (58). Let $(x_0, t_0/2)$ be the unique intersection point of $\gamma_n^+(a)$ with the horizontal line $\{(x, t_0/2) : x \in \mathbf{R}\}$. Then by Lemma 7 $x_0 > a$, and, by decreasing if necessary t_0, we achieve that $x_0 < a + \epsilon/3$. Now we put $\alpha = x_0$. Using (63) and the fact that f and ηf have no sign changes on $(a, a+\epsilon)$ we obtain

$$s((\alpha, \beta), f) = s((a, \beta), f) = s((-\infty, \infty), \eta f), \tag{66}$$

which is similar to (56). Now we repeat the argument from Case 1. Consider the functions $g_{t,\tau}$ defined as in (59). Using Lemma 11 and (66), we obtain

$$s((\alpha, \beta), f) \geq s((\alpha, \beta), g_{0,\tau}), \quad \tau \in (0, t_0/2), \tag{67}$$

as in (61). Now, in view of (57), (58), (64) and (65) the function $u_\tau(x, t) = g_{t,\tau}(x)$ satisfies

$$|u_\tau(x, t)| \geq \delta/2, \quad \text{for all} \quad \tau \in (0, t_0/2)$$

on the lateral sides of the quadrilateral Q whose sides are $(a, \beta) \times \{0\}$, $(\alpha, \beta) \times \{t_0/2\}$, the vertical segment $[(\beta, 0), (\beta, t_0/2)]$ and the piece of the curve $\gamma_n^+(a)$ from the point $(a, 0)$ to the point $(\alpha, t_0/2)$. Applying Lemma 12 to this quadrilateral Q and function $u_\tau(x, t)$ we obtain

$$s((\alpha, \beta), g_{t,\tau}) \leq s((a, \beta), g_{0,\tau}) + 4.$$

Using (67), and passing to the limit when $\tau \to 0$ and taking into the account that the number of sign changes does not increase in the limit, we obtain

$$s((\alpha, \beta), K_t * f) = s((\alpha, \beta), g_{t,0}) \leq s((\alpha, \beta), f) + 4.$$

The remaining two cases are completely similar. □

Completion of the proof of Theorem 1. It remains to put the pieces together. Let f be a measure satisfying the conditions of Theorem 1. Fix $x_0 \in \mathbf{R}$, a jump point of $s(r, f)$. We may assume without loss of generality that $x_0 < 0$. Suppose, by contradiction, that for some $\eta \in (0, \min\{a/4, 1\})$ there exists a sequence $x_k \to +\infty$ such that

$$s((x_0, x_k), f) < (a - 4\eta)x_k/\pi - 8. \tag{68}$$

We apply the theorem of Beurling and Malliavin to find a multiplier g of type η, such that $g(x) \geq 0$ for $x \in \mathbf{R}$. Then gf is a measure of finite variation which has at most as many sign changes as f has on every interval, and according to Lemma 3 gf has a spectral gap $(-a+\eta, a-\eta)$.

For $t>0$, let $(fg)_t = K_t * (fg)$ and let

$$(fg)_t = h_t + \overline{h_t}$$

be the decomposition as in Lemma 1. We apply Lemma 5 to find $r_0>0$ and $t_0>0$ such that (40) holds with $\epsilon = \eta^2/8$. Choose $r_1 > 2r_0$ so that

$$(a-2\eta)r/\pi - C_1(2r_0 + r_0^{-1})/\pi - 1 > (a-3\eta)r/\pi, \quad \text{for} \quad r > r_1, \tag{69}$$

where C_1 is the upper bound for $J(\log|h_t|)$ from Lemma 5. Then Proposition 3 applied to $(fg)_t$ together with (69) implies that

$$s(r,(fg)_t) > (a-3\eta)r/\pi, \quad t \in (0,t_0),\ r > r_1, \tag{70}$$

if all zeros of $(fg)_t$ are simple. Now suppose that $x_k > r_1+1$, and let x_k' be the smallest jump point of $s(r,f)$ such that $x_k' \geq x_k$. Let $\epsilon \in (0,1)$ be so small that there are no other jump points in the ϵ-neighborhood of x_k'. Applying Proposition 4 to the interval (x_0, x_k'), we obtain α and β in ϵ-neighborhoods of x_0 and x_k', respectively, such that

$$s((\alpha,\beta),(fg)_t) \leq s((\alpha,\beta),f) + 4, \tag{71}$$

for $t \in (0,t_k)$ with some $t_k \in (0,t_0) > 0$. We can choose this t so that all zeros of $(fg)_t$ are simple. Now we recall that jumps of $s(r,f)$ at x_k' are at most 2, so $s((\alpha,\beta),f) \leq s((x_0,x_k),f)+4$, and we obtain using (71) and (68)

$$\begin{aligned} s(\beta,(fg)_t) &\leq s((\alpha,\beta),(fg)_t) \leq s((\alpha,\beta),f)+4 \\ &\leq s((x_0,x_k),f)+8 < (a-4\eta)x_k/\pi \\ &< (a-3\eta)\beta/\pi. \end{aligned}$$

This contradicts (70), and proves the theorem. □

8 Proof of Theorem 2

Now we proceed to the proof of Theorem 2. We represent f as a sum

$$f = f_+ + f_-,$$

where $\operatorname{supp} f_+ \subset [0, \infty)$ and $\operatorname{supp} f_- \subset (-\infty, 0]$. Then for every $t > 0$ we have

$$K_t * f = K_t * f_+ + K_t * f_-, \tag{72}$$

where K_t is the heat kernel. All three terms in (72) are real analytic on the real line for every $t > 0$. Moreover,

$$(K_t * f_-)(s) = O(e^{-s^2}), \ s \to +\infty, \tag{73}$$

and

$$(K_t * f_+)(s) = O(e^{-s^2}), \ s \to -\infty.$$

In addition, (3) implies that

$$(K_t * f_\pm)(s) = O(e^{\lambda|s|}), \quad s \to \infty \tag{74}$$

for every $\lambda > 0$. So the following integrals are absolutely convergent and analytic in their half-planes:

$$G_t^+(z) = \int (K_t * f_-)(s) e^{-isz} ds = \hat{K}_t(z) F^+(z), \quad \operatorname{Im} z > 0, \tag{75}$$

and

$$G_t^-(z) = -\int (K_t * f_+)(s) e^{-isz} ds = \hat{K}_t(z) F^-(z), \quad \operatorname{Im} z < 0. \tag{76}$$

Here (F^+, F^-) is the generalized Fourier transform of f as in (4).

Notice that for every fixed $y > 0$, $G_t^+(. + iy)$ is the Fourier transform of the function

$$s \mapsto (K_t * f_-)(s) \exp(sy),$$

which is infinitely differentiable and decreases exponentially fast as $s \to \infty$. This follows from (73) and (74). We conclude that

$$G_t^+(x + iy) = O(x^{-N}), \quad x \to \infty$$

for every $y > 0$ and every $N > 0$. Similar remark applies to G_t^- for every fixed $y < 0$.

As

$$\hat{K}_t(z) = \exp(-z^2 t/4) \tag{77}$$

is entire, we conclude that $G_t^{\pm}$ are analytic continuations of each other through $(-a, a)$. As $G_t^{\pm}$ decrease faster than any power on horizontal lines, and we can use the Fourier inversion formula:

$$(K_t * f_-)(s) = \frac{1}{2\pi} \int_{\mathrm{Im}\, z=y} G_t^+(z) e^{isz} dz, \quad s \in \mathbf{R}, \quad y > 0,$$

and

$$(K_t * f_+)(s) = -\frac{1}{2\pi} \int_{\mathrm{Im}\, z=y} G_t^-(z) e^{isz} dz, \quad s \in \mathbf{R}, \quad y < 0.$$

Both integrals are absolutely convergent. We fix arbitrary $\epsilon \in (0, a)$, and choose $y = \epsilon$ in the first integral and $y = -\epsilon$ in the second one. Adding them and using (72), we write the result as

$$(K_t * f)(s) = \frac{1}{2\pi} \int_{\gamma(\epsilon)} G_t(z) e^{isz} dz, \quad s \in \mathbf{R}, \tag{78}$$

where $G_t = (G_t^+, G_t^-)$ and $\gamma(\epsilon)$ is the contour shown in Fig. 3.

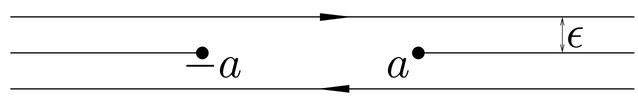


Figure 3: The path γ_ϵ

The integrand of (78) is holomorphic in

$$\mathbf{C} \backslash ((-\infty, a] \cup [a, \infty))$$

so we can deform the contour $\gamma(\epsilon)$ into the sum $\gamma(\epsilon) = \gamma_+(\epsilon) + \gamma_-(\epsilon)$ as in Fig. 4.

Figure 4: The path γ_ϵ splitted into $\gamma_+(\epsilon)$ and $\gamma_-(\epsilon)$

Now we define the function h_t holomorphic in the upper half-plane:

$$h_t(\zeta) = \frac{1}{2\pi} \int_{\gamma_+(\epsilon)} G_t(z) e^{i\zeta z} dz, \quad \operatorname{Im} \zeta > 0. \tag{79}$$

In view of (75), (76) and (77) we can pass to the limit when $t \to 0$ in (79) and obtain

$$h_t(\zeta) \to h_0(\zeta) := \frac{1}{2\pi} \int_{\gamma_+(\epsilon)} F(z) e^{i\zeta z} dz, \quad \operatorname{Im} \zeta > 0,$$

uniformly on compact subsets of the upper ζ-half-plane. Here $F = (F^+, F^-)$, and the last integral is absolutely convergent because of the exponential factor in it. Let $\sigma_0 \in (0,1)$, be such that $h_0(i\sigma_0) \neq 0$. Then we have

$$|h_t(i\sigma_0)| \geq \delta > 0, \quad t \in (0,1) \tag{80}$$

with δ independent of t.

As f is real, we have $G_t(-\bar{z}) = \bar{G_t}(z)$, and thus

$$\overline{h_t}(\zeta) = \overline{h_t(\overline{\zeta})} = \frac{1}{2\pi} \int_{\gamma_-(\epsilon)} G_t(z) e^{i\zeta z} dz, \quad \operatorname{Im} \zeta < 0. \tag{81}$$

Both integrals (79) and (81) still converge for real ζ. Adding them for $\zeta = s \in \mathbf{R}$ and comparing the result with (78) we conclude that

$$(K_t * f)(s) = h_t(s) + \overline{h_t}(s), \quad s \in \mathbf{R}. \tag{82}$$

Now we estimate h_t in the upper half-plane. It is important that our estimates will be independent of t, for $t \in (0,1)$. We denote

$$B(\epsilon) = \sup\{|G_t(x+iy)| : x+iy \in \gamma_+(\epsilon),\, t \in (0,1)\}.$$

It follows from (75) and (76) that $B(\epsilon) < \infty$ for every $\epsilon \in (0,a)$, because the generalized Fourier transform of a measure is bounded on every line $\operatorname{Im} \zeta = \text{const} \neq 0$.

Now we put $z = x + iy$ and estimate the integral in (79):

$$|h_t(s+i\sigma)| \leq \frac{B(\epsilon)}{2\pi} \int_{\gamma_+(\epsilon)} e^{-(\sigma x + sy)} |dz| \leq \frac{B(\epsilon)}{\sigma} e^{-(a-\epsilon)\sigma + \epsilon |s|}, \quad \sigma > 0. \tag{83}$$

Now we estimate the number of sign changes of $f_t = K_t * f$ from below.

Lemma 13 *Let f_t be a real analytic function with only simple seros on the real line, and assume that*

$$f_t(x) = h_t(x) + \overline{h_t}(x), \quad x \in \mathbf{R},$$

where h_t is holomorphic in the closed upper half-plane, and satisfies (83) *and* (80)*. Then*

$$\int_{\sigma_0}^{r} \frac{s(x, f_t) + s(-x, f_t)}{x} dx \geq (2a/\pi - 2\epsilon)r - \log B(\epsilon) + C, \tag{84}$$

where C is a constant that depends only on δ and σ_0 in (80)*.*

Proof. To simplify our formulas, we do not write the subscript t in this proof. Out estimates do not depend on t. Consider the following function:

$$u(z) = \begin{cases} \log|h(z)|, & \operatorname{Im} z \geq 0, \\ \\ \log|\overline{h}(z)|, & \operatorname{Im} z < 0. \end{cases}$$

It is continuous, delta-subharmonic in $\mathbf{C}$, and subharmonic in the complement of the real axis. Inequality (80) implies

$$u(i\sigma_0) \geq \log \delta > -\infty. \tag{85}$$

The restriction $d\mu$ of the Riesz charge of u on the real axis is a sum of absolutely continuous and discrete components:

$$d\mu = \frac{1}{\pi} \frac{\partial u}{\partial y}(x + i0)dx + d\mu_0, \tag{86}$$

where the discrete component $d\mu_0$ is the counting measure of real zeros of h. These zeros are simple by our assumptions. As in the proof of Theorem 1 in the real analytic case, we study a harmonic conjugate v to u in a "half-neighborhood" of the real line, that is in some region of the form

$$U = \{x + iy : 0 < y < \delta(x)\},$$

where $\delta > 0$ is a continuous function. As f is real analytic, h is also real analytic, so the zeros of h do not accumulate to the real axis, and u is harmonic near the real axis. Thus a harmonic conjugate v is defined in some region U. We have

$$v(x) = \arg h(x + i0),$$

the branch obtained as the limit from the upper half-plane. At the points where $h(x) \neq 0$, he Cauchy–Riemann equations give $\partial v/\partial x = -\partial u/\partial y$. If $h(x_0) = 0$ then $v(x_0 + 0) - v(x_0 - 0) = -\pi$, while $d\mu_0$ has an atom of mass $+1$ at $x - 0$. So we obtain from (86)

$$v(x) - v(-x) = -\pi \int_{-x}^{x} d\mu =: -\pi n(x). \tag{87}$$

Now we apply Jensen's formula to the disc $\{z : |z - \zeta_0| \leq r\}$, where $\zeta_0 = i\sigma_0$ is the point from (85). We set $w(x) = \sqrt{x^2 - \sigma_0^2}$, so that

$$w(x) < x \quad \text{for} \quad x > \sigma_0. \tag{88}$$

Our function u is subharmonic in $\mathbf{C}\backslash\mathbf{R}$, so Jensen's formula implies

$$\int_{\sigma_0}^{r} \frac{n(w(x))}{x} dx \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\zeta_0 + re^{i\theta}) d\theta - u(\zeta_0), \tag{89}$$

where n was defined in (87). Using (87), (89), and estimating the integral in the right hand side of (89) with the help of (83), we obtain

$$\frac{1}{\pi} \int_{\sigma_0}^{r} \frac{v(w(x)) - v(-w(x))}{x} dx = - \int_{\sigma_0}^{r} \frac{n(w(x))}{x} dx \tag{90}$$

$$\geq -\frac{1}{2\pi} \int_{-\pi}^{\pi} u(\zeta_0 + re^{i\theta}) d\theta + u(\zeta_0) \tag{91}$$

$$\geq \frac{(a - \epsilon)r}{2\pi} \int_{-\pi}^{\pi} |\sin\theta| d\theta - \epsilon r - \log B(\epsilon) + \log\delta + C_1 \tag{92}$$

$$= 2(a - \epsilon)r/\pi - \epsilon r - \log B(\epsilon) + C, \tag{93}$$

where C_1 is an absolute constant, and $C = \log\delta + C_1$ depends only on δ. Recalling that $s(r, f) + s(-r, f) \geq (v(r) - v(-r))/\pi - 1$ (every half-turn of $h(x)$ around zero yields at least one sign change of $f = 2\Re h$), using monotonicity of $s(r, f)$ and (88), we obtain

$$\int_{\sigma_0}^{r} (s(x, f) + s(-x, f)) \frac{dx}{x} \geq \int_{\sigma_0}^{r} (s(w(x), f) + s(-w(x), f)) \frac{dx}{x}$$

$$\geq \frac{1}{\pi} \int_{\sigma_0}^{r} (v(w(x)) - v(-w(x))) \frac{dx}{x} - \log r + \log\sigma_0$$

$$\geq 2(a - \epsilon)r/\pi - \epsilon r - \log B(\epsilon) + C - \log r + \log\sigma_0.$$

This proves the Lemma. □

Now we complete the proof of Theorem 2.

Let us first fix $R > 1$. Let $r \in (0, R)$. Let (x, y) be the minimal interval whose endpoints are jump points of $s(r, f)$ and which contains $(-r, r)$. Then

$$s((-r, r), f) = s((x, y), f). \tag{94}$$

Applying Proposition 4 to the interval (x, y) with $\epsilon \in (0, 1)$, we obtain the points α and β in ϵ-neighborhoods of x and y respectively, such that

$$s((\alpha, \beta), f_t) \leq s((\alpha, \beta), f) + 4, \tag{95}$$

for $t \in (0, t_0(R)$. Assuming that ϵ is so small that there are no other jump points of $s(r, f)$ in the ϵ-neighborhoods of x and y, and taking into account that the magnitude of jumps is at most two, we obtain from (94) and (95) that

$$\begin{aligned} s((-r, r)), f) &\geq s((\alpha, \beta), f) - 4 && (96)\\ &\geq s((\alpha, \beta), f_t) - 8 \geq s((-r + 1, r - 1), f_t) - 8, && (97) \end{aligned}$$

or

$$s((-r, r), f_t) \leq s(-r - 1, r + 1), f) + 8, \tag{98}$$

for every $r \in (0, R)$ and every $t \in (0, t_0)$. Dividing (98) by r and integrating from σ_0 to R gives

$$S(R, f_t) \leq \int_{\sigma_0+1}^{R+1} s((-r, r), f) \frac{dr}{r - 1} + 8 \log(R/\sigma_0).$$

Now, for an arbitrary given $\epsilon > 0$, we have $(r-1)^{-1} < (1+\epsilon)r^{-1}$ for $r > \epsilon^{-1}$. Using this to estimate the integral in the RHS, we obtain

$$S(R, f_t) \leq (1 + \epsilon)S(R + 1, f) + C(\epsilon) + 8 \log R.$$

Comparing this with (84) gives the result.

The second assertion of Theorem 2 is proved similarly, using Carleman's formula [25, p. 187] instead of Jensen's formula. □

The above proof can be generalized to a class of distributions. The appropriate class is $(S_1^\beta)'$, $\beta > 2$ of Gelfand and Shilov [14].

We recall that the space of test functions S_1^β consists of infinitely differentiable functions ϕ satisfying

$$\sup_x |x^k \phi^{(q)}(x)| \leq C A^k B^q k^k q^{\beta q},$$

where the constants A, B, C may depend on ϕ. This is a union over all positive A and B of countably normed spaces $S_{1,A}^{\beta,B}$ with the norms

$$\|\phi\|_{m,n} = \sup_{x,k,q} \frac{|x^k \phi^{(q)}(x)|}{(A+1/m)^k (B+1/n)^q k^k q^{q\beta}}, \quad m,n = 1,2,\ldots. \tag{99}$$

All spaces S_1^β are contained in the Schwartz space $\mathscr{S}$, and convergence in S_1^β implies convergence in $\mathscr{S}$. Spaces S_1^β contain functions with bounded support if and only if $\beta > 1$. In this case, for every interval I, the functions of the class S_1^β with bounded support on I are dense in the space of all continuous functions with bounded support on I. All functions $\phi \in S_1^\beta$ satisfy

$$\sup_{x \in \mathbf{R}} |\phi^{(q)}(x)| \exp(-\lambda |x|) < \infty \quad \text{for all} \quad q \geq 0 \quad \text{and some} \quad \lambda > 0.$$

Let $(S_1^\beta)'$ be the space of all continuous linear functionals on S_1^β. Let η be an sufficiently smooth function, $\eta(x) = 0$ for $x \leq -1$ and $\eta(x) + \eta(-x) \equiv 1$. Then the functions

$$F^+(\zeta) = (f(x), \eta(-x) e^{-ix\zeta}) \quad \text{and} \quad F^-(\zeta) = -(f(x), \eta(x) e^{-ix\zeta})$$

are analytic in the upper and lower half-planes, respectively, and the *generalized Fourier transform* is defined as a hyperfunction (F^+, F^-). Changing η results in addition of the same entire function to F^+ and F^-, so our definition is independent of the choice of η. Then the spectrum and the spectral gap of a distribution of the class $(S_1^\beta)'$ is defined as in the Introduction, in terms of the generalized Fourier transform.

We estimate this generalized Fourier transform on horizontal lines.

Lemma 14 *Let f be a distribution of the class $(S_1^\beta)'$ with $\beta > 1$. Then one can choose η in the definition of the generalized Fourier transform $F = (F^+, F^-)$ such that for every fixed $y \in \mathbf{R} \backslash \{0\}$ and every $\beta' \in (1, \beta)$,*

$$F(x+iy) = O(\exp |x|^{1/\beta'}), \quad x \to \infty.$$

Proof. Fix a number $\gamma \in (1,\beta)$. Then there exists a smooth function η, as in the definition of the generalized Fourier transform, with the additional property

$$\sup_{x\in\mathbf{R}} |\eta^{(j)}(x)| \leq Cj^{\gamma j}, \quad j = 0, 1 \ldots, \tag{100}$$

where we use the convention that $0^0 = 1$.

Let us assume, for example, that $y < 0$, and estimate F^-. The estimate of F^+ is similar. We have

$$|F^-(z)| = |(f, \eta(t)e^{-itz})| \leq C\|\eta(t)e^{-itz}\|,$$

where $\|.\|$ is one of the norms (99) and C is a constant depending of f and η. Straightforward estimation of the norm using (100) gives the Lemma. $\Box$

Now we state a version of Theorem 2 for distributions.

Theorem 2′ *Let $f \in (S_1^\beta)'$, $\beta > 2$ be a distribution with a spectral gap $(-a, a)$. Then the conclusions of Theorem 2 hold.*

We only explain modifications needed in the proof. The definition of $B(\epsilon)$ has to be modified as follows. Fix $\alpha \in (1,\beta)$ and put

$$B(\epsilon) = \sup\{\exp(-|x|^{1/\alpha})\, |G_t(x+iy)| : x+iy \in \gamma_+(\epsilon),\ t \in (0,1)\}.$$

It follows from Lemma 14 that $B(\epsilon) < \infty$. Now we put $z = x + iy$ and estimate the integral in (79):

$$|h_t(s+i\sigma)| \leq \frac{B(\epsilon)}{2\pi} \int_{\gamma_+(\epsilon)} \exp(-\sigma x - sy + x^{1/\alpha})|dz|.$$

Applying Laplace's method to estimate the last integral, we obtain

$$|h_t(s+i\sigma)| \leq CB(\epsilon)\exp\{-(a-\epsilon)\sigma + C_1\sigma^{1/(1-\alpha)} + \epsilon|s|\}, \quad \sigma > 0. \tag{101}$$

This estimate replaces (83). The main point is that $\exp\sigma^{1/(1-\alpha)}$ has integrable logarithm near $\sigma = 0$ when $\alpha > 2$, so the proof of Lemma 13 needs no modification. A serious modification is needed in our final argument where we substantially used the fact that for measures f, the magnitudes of jumps of $s(r, f)$ are at most 2. This parts needs no modification for distributions of finite order. For the general case, the following generalization of Lemma 7 is needed.

Lemma 15 *Let f be a distribution with support on $[x_0, +\infty)$, with finitely many changes of sign on every finite interval, and such that the convolution $u(x,t) = (K_t * f)(x)$ is well defined. Assume that f has a change of sign at x_0, and that u is unbounded in every neighborhood of x_0. Then there exists a curve γ in the upper half-plane except one endpoint at x_0, and a number $t_0 > 0$ such that u is bounded from above or from below in the region whose boundary consists of the ray $\{(x,0) : x \leq x_0\}$, the curve γ and a ray $\{(x,t_0) : x < x_1\}$, where $(x_1,t_0) \in \gamma$, and in addition*

$$\inf\{|u(x,t)| : (x,t) \in \gamma\} > 0.$$

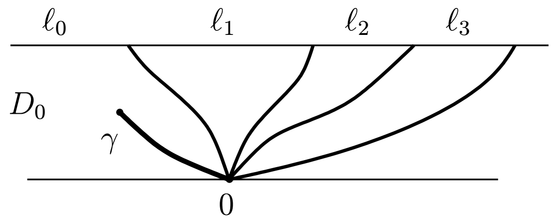
Proof. We assume without loss of generality that $x_0 = 0$. It is enough to prove the lemma in the case that the only jump point of $s(r,f)$ occurs at 0. So we assume without loss of generality that the restriction of f on $(0,\infty)$ is a positive measure. If this positive measure has no concentration points accumulating to 0, the lemma follows from Lemma 7. So we assume that there are concentration points of f arbitrarily close to 0. Then it follows from Lemma 8 that there exists a neighborhood U of $(0,+\infty)$ such that $u > 0$ in U.

It follows from Lemma 11 that the number of sign changes of f_t is bounded from above independently of t. On the other hand, for sufficiently small t, the functions f_t have at least one change of sign; this follows from our assumption that 0 is a jump point of $s(r,f)$. Moreover, by Lemma 11, the number of sign changes of f_t is a decreasing function of t, so we can choose $T > 0$ so small that

$$\text{the number of sign changes of } f_t \text{ is constant for } t \in (0,T). \tag{102}$$

Consider the strip $P = \{(x,t) : 0 < t < T\}$ and the temperature $u(x,t) = (K_t * f)(t)$ in this strip. Let $\ell_0, \ell_1, \ldots, \ell_n$ be the maximal intervals of sign constancy of f_T, enumerated left to right, $n \geq 1$. To each of these intervals ℓ_j corresponds a unique component D_j of the set $\{(x,t) : u(x,t) \neq 0\}$ such that $\ell_j \subset \partial D_j$. This was established in the proof of Lemma 12. By the Maximum Principle (Lemma 8), each ∂D_j intersects the $x-$axis, so our assumptions about f imply that the sign of u in D_n is $+1$. Then it alternates, so the sign of u in D_{n-j} is $(-1)^j$.

We claim that for every $j < n$, the boundary of D_j does not intersect the open ray $(0,+\infty)$. Indeed, this is enough to prove for D_{n-1} as it blocks all



other domains from the right. But the sign of u in D_{n-1} is negative, while there exists a neighborhood U of the open positive ray such that $u > 0$ in U, so ∂D_{n-1} cannot intersect $(0, \infty)$, and this proves our claim.

In particular, $\partial D_0 \cap \{(x,t) : t = 0\} \subset (-\infty, 0]$. Moreover, Lemma 8 implies that $0 \in \partial D_0$. It follows from (102) that the intersection of D_0 with every horizontal line $t =$ const consists of exactly one interval, so, in particular,

$$\partial D_0 \cap \{(x,t) : t = 0\} = (-\infty, 0]. \tag{103}$$

By Lemma 9, u is unbounded in D_0. Let G_1 be the component of the set

$$\{(x,t) \in D_0 : |u(x,t)| > 1\}.$$

It is clear that $\partial G_1 \cap \{(x,t) : t = 0\} = \{0\}$, and that u is unbounded in G_1. Now we construct a sequence of domains $G_1 \supset G_2 \supset \ldots$, so that G_j is a component of the set

$$\{(x,t) \in D_0 : |u(x,t)| > j\}.$$

For the same reasons as above, $\partial G_j \cap \{(x,t) : t = 0\} = \{0\}$, for every $j = 1, 2, \ldots$. Now we choose points $s_j \in G_j \backslash G_{j-1}$ and connect each pair (s_j, s_{j+1}) by a curve $\gamma_j \subset G_j$. The union γ of these curves satisfies all requirements of the lemma. $\Box$

This lemma permits to prove a more precise version of Proposition 4: one can additionally insure that $(\alpha, \beta) \subset (a, b)$. This permits to modify the estimates (96), (97) in the following way:

$$s((-r,r)),f) = s((\alpha,\beta),f) \geq s((\alpha,\beta),f_t) - 4 \geq s((-r+1,r-1),f_t) - 4.$$

The rest of the proof of Theorem 2 remains unchanged.

9 Limit sets of entire functions

The theorem of Cartwright and Levinson mentioned in section 2 shows that constructing an example of an efet whose indicator diagram is an interval of the imaginary axis, and which does not have completely regular growth, may be a non-trivial task. First such example was constructed by Redheffer [36], see also [37, 20]. An application of these examples was to show that Titchmarsh's theorem on the support of convolution fails for hyperfunctions with bounded support. However, all these examples are still too regular for our purposes, and we need the theory of limit sets, which generalizes the theory of completely regular growth. The theory of limit sets of entire functions was developed by Azarin, Giner [2, 3, 4], Hörmander and Sigurdsson [18]. Here we collect the necessary facts from this theory.

Let U^* be the set of all subharmonic functions in the plane satisfying

$$\limsup_{|z|\to\infty} |z|^{-1}u(z) < \infty,$$

with the induced topology from the space of Schwartz distributions $\mathscr{D}'(\mathbf{C})$, and

$$U(\sigma) = \{u \in U^* : u(0) = 0, \sup_{z\in\mathbf{C}} |z|^{-1}u(z) \leq \sigma\},$$

for $\sigma > 0$. We recall that on the set SH of all subharmonic functions in $\mathbf{C}$, topologies induced from $\mathscr{D}'(C)$ and from $L^1_{\mathrm{loc}}(\mathbf{C})$ coincide, and SH is closed. This follows from theorems 4.1.8 and 4.1.9 in [17, Vol. I]. So SH is a metric space.

We denote $U = \cup_{\sigma>0}U(\sigma)$. A one-parametric group A of operators

$$(A_tu)(z) = t^{-1}u(tz), \quad t > 0,$$

acts on U^*. The sets $U(\sigma)$ are A-invariant.

For a function $u \in U^*$ we define the *limit set* $\mathrm{Fr}\,[u] = \mathrm{Fr}_\infty[u]$ as the set of all limits

$$\lim_{n\to\infty} A_{t_n}u \quad \text{for} \quad t_n \to \infty.$$

Similarly, $\mathrm{Fr}_0[u]$ is defined for $u \in U$, using sequences $t_n \to 0$. Each limit set $\mathrm{Fr}_\infty[u]$ or $\mathrm{Fr}_0[u]$ is a closed connected A-invariant subset of $U(\sigma)$ for some $\sigma > 0$. If f is an efet then $\log|f| \in U^*$, and we define the *limit set of f* as $\mathrm{Fr}\,[\log|f|]$. For every limit set $\mathrm{Fr}\,[u]$, the function

$$v(z) = \sup\{w(z) : w \in \mathrm{Fr}\,[u]\} \tag{104}$$

is A-invariant and subharmonic. All such functions have the form

$$v(re^{i\theta}) = rh(\theta), \quad \text{where} \quad h'' + h \geq 0, \tag{105}$$

that is $h'' + h$ is a positive measure. Functions h with this property are called *trigonometrically convex.* The function h defined by (104) and (105) is called the *indicator* of u. If f is an efet and h the indicator of $\log|f|$, then h coincides with the classical Phragmén–Lindelöf indicator of f. The *indicator diagram* is the closed convex set in the plane whose support function is h. Pólya's theorem says that the spectrum of f is obtained from its indicator diagram by rotation by $\pi/2$.

Criteria for a subset $\mathcal{F} \subset U$ to be a limit set of some function $u \in U^*$ were found in [3] and [18]. The following result is from [3] (see also [4]):

Proposition 5 *Fix $\sigma > 0$. For a closed connected A-invariant subset $\mathcal{F} \subset U(\sigma)$, the following conditions are equivalent:*
a) $\mathcal{F} = \mathrm{Fr}\,[u]$ for some $u \in U^*$,
b) $\mathcal{F} = \mathrm{Fr}\,[\log|f|]$ for some efet f, and
c) There exists a piecewise continuous map

$$\mathbf{R}_{>0} \to U(\sigma), \quad t \mapsto v_t$$

with the properties

$$\mathrm{dist}\,(A_\tau v_t,\, v_{\tau t}) \to 0, \quad t \to \infty, \quad \forall \tau > 0$$

and

$$\mathrm{clos}\,\{v_t : t \in (t_0, \infty)\} = \mathcal{F}, \quad \forall t_0 > 0.$$

Here are some simple examples of limit sets derived from Proposition 5.

1. One-point limit set. Its only element has to be of the form (105). This characterizes completely regular growth in the sense of Levin–Pfluger.

2. One periodic orbit. Let u be a subharmonic function with the property that $A_T u = u$ for some $T \neq 1$. Then

$$\{A_t u : 1 \leq t \leq T\}$$

is a limit set. One can show that in this case the indicator diagram cannot be a non-degenerate interval of the imaginary axis, so this type of function is not appropriate for our purposes.

3. The closure of a single orbit,

$$\{A_t u : 0 < t < \infty\} \cup \mathrm{Fr}_0[u] \cup \mathrm{Fr}_\infty[u], \quad \text{where} \quad u \in U(\sigma),$$

is a limit set if and only if

$$\mathrm{Fr}_0[u] \cap \mathrm{Fr}_\infty[u] \neq \emptyset.$$

Again, in this case the indicator diagram cannot be a non-degenerate interval of the imaginary axis.

4. An interval. If u_0 and u_1 are two functions of the form (105) then the set

$$\{tu_0 + (1-t)u_1 : 0 \leq t \leq 1\}$$

is a limit set.

Examples in [20] are of this sort. The efet constructed in [20] has indicator diagram $[-ib, ib]$ and the *lower* density of zeros is strictly less than $2b/\pi$. We need an example of an even efet with the indicator diagram $[-ib, ib]$ and the *upper* density of zeros strictly greater that $2b/\pi$. To achieve this we combine the last two examples.

Lemma 16 *Let u be a function in U with the properties*

$$\mathrm{Fr}_0[u] = \{u_0\} \quad \textit{and} \quad \mathrm{Fr}_\infty[u] = \{u_1\}.$$

Then

$$\mathcal{F} = \{A_t u : 0 < t < \infty\} \cup \{tu_0 + (1-t)u_1 : 0 \leq t \leq 1\} \tag{106}$$

is a limit set.

Proof. This easily follows from the general criterion in Proposition 5. Fix a sequence of positive numbers with the property $r_{k+1}/r_k \to \infty,\ k \to \infty$.

If $k = n^2$ for a positive integer n, we set $s_k = \sqrt{r_k r_{k+1}}$, and

$$v_t = A_{t/s_k} u, \quad r_k \leq t < r_{k+1}.$$

If $k = n^2 + j$, where $1 \leq j \leq 2n$, we define

$$v_t = (j/2n)u_0 + ((2n-j)/2n)u_1, \quad r_k \leq t < r_{k+1}.$$

Then it is easy to verify that v_t satisfies condition c) of Proposition 5 with $\mathcal{F}$ as in (106). $\square$

Now we describe the relation between the limit set and the distribution of zeros of an efet. Consider the set of all measures in $\mathbf{C}$. The analog of operators A_t for measures is

$$(B_t\mu)(E) = t^{-1}\mu(tE), \quad \text{for Borel sets} \quad E \subset \mathbf{C}.$$

The Laplace operator $(2\pi)^{-1}\Delta$ semi-conjugates A_t and B_t:

$$\Delta A_t = B_t \Delta. \tag{107}$$

We denote by V^* the set of all positive measures μ, which satisfy

$$\limsup_{r\to\infty} r^{-1}\mu(D(r)) < \infty,$$

where $D(r) = \{z \in \mathbf{C} : |z| \leq r\},\ r \geq 0$. We also define the subsets

$$V(\sigma) = \{\mu \in V^* : \mu(D(r)) \leq r\sigma,\ 0 < r < \infty\}, \quad \sigma > 0,$$

and $V = \cup_{\sigma>0}V(\sigma)$. The Laplace operator is continuous in $\mathscr{D}'(\mathbf{C})$ and sends U to V (however, this map is not surjective, and the image of $U(\sigma)$ is not equal to $V(\sigma')$ for any $\sigma' > 0$). Given a measure $\mu \in V^*$, we define the limit set $\mathrm{Fr}\,[\mu]$ as the set of all limits in $\mathscr{D}'(\mathbf{C})$

$$\lim_{n\to\infty} B_{t_n}\mu \quad \text{for} \quad t_n \to \infty.$$

It follows from (107) that for every $u \in U^*$ we have

$$(2\pi)^{-1}\Delta\left(\mathrm{Fr}\,[u]\right) = \mathrm{Fr}\,\left[(2\pi)^{-1}\Delta u\right]. \tag{108}$$

If f is entire, then $(2\pi)^{-1}\Delta \log|f|$ is the counting measure of zeros of f. So the asymptotic distribution of zeros is reflected in the Riesz measures of the elements of the limit set. Let us make this more precise. Two measures in U^* are called equivalent if

$$B_t(\mu_1 - \mu_2) \to 0 \quad \text{as} \quad t \to \infty.$$

This implies $\mathrm{Fr}\,[\mu_1] = \mathrm{Fr}\,[\mu_2]$. Let $T : \mathbf{C} \to \mathbf{C}$ be a map with the property

$$T(z) - z = o(z), \quad z \to \infty. \tag{109}$$

We recall that push-forward $T_*\mu$ of a measure by T is defined by $(T_*\mu)(E) = \mu(T^{-1}(E))$. If $\mu \in V^*$, and a map T satisfies (109), then $T_*\mu$ is equivalent to μ. For each $\mu \in V^*$ one can construct a map T with the property (109) such that $T_*\mu$ is a counting measure of a divisor in $\mathbf{C}$. This explains the implication a)$\to$b) in Proposition 5.

Lemma 17 *Let μ be a measure in V^*. Suppose that all measures in* $\mathrm{Fr}\,[\mu]$ *are supported on the real line and have the form $d(x)dx$ where $d(x) < 1$. Then there exists a measure μ_1 equivalent to μ, which is supported on the integers, and $\mu_1(n) \in \{0, 1\}$ for each integer n.*

Proof. First we project our measure μ by the map

$$T(re^{i\theta}) = \begin{cases} r, & |\theta| < \pi/2, \\ -r, & |\theta - \pi| \le \pi/2. \end{cases}$$

This map does not satisfy (109) but it is easy to see that $\mu_0 = T_*\mu \sim \mu$ for a measure μ satisfying the conditions of Lemma 17.

Second, let F_0 be the distribution function of μ_0, that is $\mu_0 = dF_0$ and $F_0(0) = 0$. Then we set $F_1(x) = [F_0([x])]$, where $[.]$ stands for the integer part, and put $\mu_1 = dF_1$. It follows from our assumptions that $F_0' < 1$, so the jumps of $F_0([x])$ are at most 1, and they occur only at integers. Thus the jumps of F_1 are all equal to one, and they occur only at integers. Now it is easy to see that $\mu_1 \sim \mu_0 \sim \mu$, so μ_1 has all required properties. $\Box$

10 Example of an efet

Here we construct Example 2 assuming, without loss of generality, that $a+b = 2\pi$. We begin with a smooth, even, non-positive function u_0 with bounded support, $u_0(0) = 0$; for example, we can take

$$u_0(x) = \begin{cases} -k(1-(x^2-1)^2)^2, & |x| \le \sqrt{2}, \\ \\ 0, & |x| > \sqrt{2}, \end{cases}$$

where $k > 0$ is a parameter to be specified later. Then we extend u_0 to $\mathbf{C}\backslash\mathbf{R}$ by Poisson's integral. The resulting function u_0 is a delta-subharmonic function in $\mathbf{C}$, whose Riesz charge is supported on $\mathbf{R}$ and has the form

$dQ_0(x) = q_0(x)dx$, where q_0 is a smooth, even, bounded function, and $Q_0(0) = 0$. So we have

$$u_0(z) = \int_0^\infty \log\left|1 - z^2/t^2\right| dQ_0(t) = \int_0^\infty \log\left|1 - z^2/t^2\right| q_0(t)dt.$$

We notice that $u_0|_{\mathbf{R}}$ is the Hilbert transform[4] of Q_0, in particular,

$$q_0(x) = Q_0'(x) = \pi^{-1}(\partial u_0/\partial y)(x + i0). \tag{110}$$

So we have

$$q_0(0) = \pi^{-1}(\partial u_0/\partial y)(0 + i0) < 0, \tag{111}$$

because 0 is a global maximum of u_0 in the plane. We put

$$-m = \min_{x\in\mathbf{R}} q_0(x) < 0 \tag{112}$$

and

$$\eta = \max_{x\geq 0} \frac{Q_0(x)}{x} > 0. \tag{113}$$

The inequality in (113) holds because $Q_0'(x) < 0$ for large positive x, in view of (110), and $Q_0(x) \to 0$ as $x \to \infty$. Now we choose and fix k so small that

$$m + \max_{x\in\mathbf{R}} q_0(x) < 1. \tag{114}$$

We define

$$Q_1(x) = Q_0(x) + mx, \quad \text{so that} \quad q_1 = Q_1' = q_0 + m \geq 0, \tag{115}$$

in view of (112), and thus the function

$$u_1(z) = u_0(z) + \pi m|\mathrm{Im}\,(z)| = \int_0^\infty \log|1 - z^2/t^2| q_1(t)dt, \tag{116}$$

is subharmonic in $\mathbf{C}$, has Riesz measure dQ_1, and belongs to the class U defined in the previous section. We have

$$\mathrm{Fr}_\infty[u_1] = \{\pi m|\mathrm{Im}\,(.)|\} \quad \text{and} \quad \mathrm{Fr}_0[u_1] = \{\pi m'|\mathrm{Im}\,(.)|\}, \tag{117}$$

[4]So, for the function u_0 written above, Q_0 can be explicitly computed.

where Im (.) is the function $z \mapsto \text{Im}\,(z)$, and

$$m' = m + q_0(0) < m. \tag{118}$$

The first formula in (117) follows from $u_0(z) \to 0$ as $z \to \infty$, while the second one and (118) follow from (111). Now by Lemma 16, the set

$$\mathcal{F} := \{A_t u_1 : t \in \mathbf{R}\} \cup \{t|\text{Im}\,(.)| : \pi m' \leq t \leq \pi m\} \subset U$$

is a limit set of an efet. Evidently,

$$\sup\{w(z) : w \in \mathcal{F}\} = \pi m |\text{Im}\,(z)|. \tag{119}$$

Let g be an entire function of exponential type m, such that

$$\text{Fr}\,[\log|g|] = \mathcal{F}.$$

According to (119), the indicator diagram of g is the interval $[-\pi m i, \pi m i]$. In other words, the Fourier transform of g is a hyperfunction supported on $[-\pi m, \pi m]$, [17, v.2, Thm.15.1.5]

In addition, we require that all zeros of g be simple and located at integers, which is possible by Lemma 17 because the Riesz measures of all elements of $\mathcal{F}$ are supported on the real line, and their densities are less then 1 in view of (114). The upper density of zeros of g on the positive ray is

$$\max_{x \geq 0} Q_1(x)/x = m + \eta. \tag{120}$$

Indeed, the limit set $\mathcal{F} = \text{Fr}\,[\log|g|]$ contains u_1. This means that there is a sequence $t_k \to \infty$ such that $B_{t_k}\Delta \log|g| \to \Delta u_1$; this follows from (108). Suppose that the maximum in (120) is attained at a point $x^* > 0$. Put $r_k = t_k x^*$ and let $n(r)$ be the number of zeros of g on the interval $[0, r]$. Then $n(r_k)/t_k = Q_1(x^*) + o(1),\ k \to \infty$, and thus $n(r)/r = Q_1(x^*)/x^* + o(1) = m + \eta + o(1),\ r \to \infty$.

Finally we set

$$f(z) = g(z)\sin \pi z.$$

Then the Fourier transform of f is a hyperfunction supported on

$$\pi[-1 - m,\ -1 + m] \cup \pi[1 - m,\ 1 + m],$$

while the sign changes occur only at those integers which are not zeros of g, that is the lower density of sign changes is at most $1 - m - \eta < 1 - m$ in view of (120). □

11 Refinement of the previous example

In this section we modify the example of the previous section so that it will have an additional property $|f(x)| \leq \exp \omega(|x|)$, for a given smooth positive weight $\omega \geq 2$ defined for $t \geq t_0 > e$, which satisfies

$$\int_{t_0}^{\infty} \frac{\omega(t)}{1+t^2} dt = \infty, \tag{121}$$

$$\frac{\omega(t)}{\log t} \quad \text{increases} \quad \text{for} \quad t > t_0, \tag{122}$$

$$\frac{\omega(t)}{t} \quad \text{decreases} \quad \text{for} \quad t > t_0, \tag{123}$$

and

$$\liminf_{t \to +\infty} \frac{\log \omega(t)}{\log t} > \frac{1}{2}. \tag{124}$$

In this section we denote by C various positive constants that depend only on the weight ω and on t_0 in (122) and (123).

1. Preliminary estimates.

The increasing function

$$H(t) = \omega(t)/\log t, \quad t \geq t_0, \tag{125}$$

has an increasing inverse $h(t)$. We define

$$B(t) = \frac{\log h(t)}{h(t)}, \quad t \geq H(t_0), \quad B(t) = \frac{\log t_0}{t_0}, \; t \in [0, H(t_0)] \tag{126}$$

Then B has the following properties which easily follow from (121), (123), and (124), respectively:

$$\int^{\infty} B(t) dt = \infty, \tag{127}$$

$$tB(t) \quad \text{decreases to} \quad 0 \quad \text{as} \quad t \to \infty, \quad t \geq H(t_0), \tag{128}$$

and

$$\liminf_{t\to\infty} \frac{\log B(t)}{\log t} \geq -2+\epsilon \tag{129}$$

for some $\epsilon > 0$. We only verify (127) leaving the other two properties to the reader:

$$\int^{\infty} B(t)dt = \int^{\infty} \frac{\log y}{y} dH(y) = \int^{\infty} \frac{(\log y - 1)}{y^2} \frac{\omega(y)}{\log y} dy = \infty.$$

Lemma 18 *(Kahane and Rubel [20, pp. 591–592]). Let*

$$\phi(u) = \frac{1}{u} \int_0^u \log \left| 1 - \frac{1}{x^2} \right| dx.$$

Then $\phi \geq 0$ *and*

$$I(r) = \int_0^{\infty} \phi(t/r) t B(t) dt = O(\omega(r)), r \to \infty.$$

Proof. We write

$$I(r) = \int_0^{H(r)} + \int_{H(r)}^{\infty},$$

where $H(r)$ is defined in (125). For the first integral we use $|tB(t)| < C$, which follows from (128), and

$$\begin{aligned} \int_0^{H(r)} \phi(t/r)dt &= r\int_0^{H(r)/r} \phi(y)dy \\ &\leq 3H(r)\log\frac{r}{H(r)} \leq 3\omega(r). \end{aligned}$$

For the second integral we use (128) again:

$$\begin{aligned} \int_{H(r)}^{\infty} \phi(t/r)tB(t)dt &\leq H(r)B(H(r)) \int_{H(r)}^{\infty} \phi(t/r)dt \\ &\leq H(r)B(H(r))r\int_0^{\infty} \phi(y)dy = C\omega(r), \end{aligned}$$

where C is an absolute constant. □

We are going to construct a subharmonic function of the form

$$u(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| q(t)dt, \tag{130}$$

where q is a positive bounded smooth function. We need estimate of the tails of this integral when $z = r$ is large. Using (124) we obtain

$$\left|\int_0^{\sqrt{r}} \log\left|1 - \frac{r^2}{t^2}\right| q(t)dt\right| \le C\int_0^{\sqrt{r}} \log\frac{r}{t}dt \le C\sqrt{r}\log r = o(\omega(r)). \tag{131}$$

Next,

$$\left|\int_{r^3}^\infty \log\left|1 - \frac{r^2}{t^2}\right| q(t)dt\right| \le C\int_{r^3}^\infty \frac{r^2}{t^2}dt = C/r = o(\omega(r)). \tag{132}$$

2. Function q_1

We recall some properties of the function q_1 from the previous section.

$$q_1 \text{ is bounded, positive and differentiable} \tag{133}$$

$$\frac{d}{dt}q_1(t) = O(t^{-2}), \quad t \to \infty, \tag{134}$$

$$\frac{d}{dt}q_1(t) = O(t), \quad t \to 0. \tag{135}$$

So,

$$\frac{d}{dt}q_1(t/R) \le C\frac{t}{R^2} < CB(t) \quad \text{if} \quad t \le \sqrt{R}. \tag{136}$$

Now we fix $K = 2/\epsilon$, where ϵ is the number from (129). Then by (134):

$$\frac{d}{dt}q_1(t/R) \le C\frac{R}{t^2} < CB(t) \quad \text{if} \quad t \ge R^K. \tag{137}$$

3. Construction of q.

After these preliminaries we begin the construction of a function q to be inserted into (130). First we define sequences $r_k \to \infty$ and $R_k > r_k^K$ such that

$$\int_{r_k^K}^{R_k} B(t)dt = q_1(r_k^{K-1}) - q_1(r_{k+1}^{-1/2}) \tag{138}$$

$$R_k \le \sqrt{r_{k+1}}, \tag{139}$$

and

$$r_{k+1} > r_k^{12K}. \tag{140}$$

Let us show that such choice of r_k and R_k is possible. We recall from section 10, that $q_1(t) \to m,\ t \to \infty$, and $q_1(t) \to m' < m,\ t \to 0$. So we can choose $r_1 > 2$ so that

$$\min\{q_1(t) : t \ge r_1^{K-1}\} > \max\{q_1(t) : 0 \le t \le r_1^{-1/2}\}.$$

Now we define r_k inductively. Suppose that r_k has been already chosen. Consider the following equation with respect to x:

$$\int_{r_k^K}^{x} B(t)dt = q_1(r_k^{K-1}) - q_1((r_k^{6K}x)^{-1}).$$

The integral increases from 0 to ∞ as x increases from r_k^K to ∞, while the right hand side varies between positive limits. So it follows from (127) that the equation has solutions $x > r_k^K$; we take the smallest solution x and denote it by R_k. Then we put $r_{k+1} = (r_k^{6K}x)^2$, so that (138), (139) and (140) are be satisfied.

Now we define our function q in the following way:

$$q(t) = q_1(t/r_k) \quad \text{for} \quad \sqrt{r_k} \le t \le r_k^K, \tag{141}$$

$$q(t) = q_1(r_k^{K-1}) - \int_{r_k^K}^{t} B(\tau)d\tau \quad \text{for} \quad r_k^K \le t \le R_k. \tag{142}$$

and

$$q(t) = q(R_k) \quad R_k \le t \le \sqrt{r_{k+1}}. \tag{143}$$

It follows from (138) that this definition gives a continuous function q.

It follows from (142) and (143) that

$$\left|\frac{dq(t)}{dt}\right| \leq B(t) \quad \text{for} \quad r_k^K \leq t \leq \sqrt{r_{k+1}}, \tag{144}$$

for every k. Furthermore, using (136) and (137), we obtain

$$\left|\frac{d}{dt}q_1(t/r_k)\right| \leq CB(t) \quad \text{for } t \text{ outside the interval} \quad [\sqrt{r_k}, r_k^K]. \tag{145}$$

3. Estimate on the real axis.

Everything is ready now for estimation from above of our function u defined in (130). We write

$$u(r) = \int_{\sqrt{r}}^{r^3} \log\left|1 - \frac{r^2}{t^2}\right| q(t)dt + v(r), \tag{146}$$

where

$$|v(r)| = o(\omega(r)), \tag{147}$$

according to (131) and (132). Now, the interval $[\sqrt{r}, r^3]$ intersects at most one of the intervals $I_k = [\sqrt{r_k}, r_k^K]$, which follows from (140). If $[\sqrt{r}, r^3] \cap I_k \neq \emptyset$, for some k, we put $I = I_k$, otherwise, $I = \emptyset$. Then we have

$$\begin{aligned} u(r) &= \int_0^\infty \log\left|1 - \frac{r^2}{t^2}\right| q(t)dt \\ &= \int_0^\infty \log\left|1 - \frac{r^2}{t^2}\right| \left(q(t) - q_1\left(\frac{t}{r_k}\right)\right) dt \qquad (148) \\ &+ \int_0^\infty \log\left|1 - \frac{r^2}{t^2}\right| q_1\left(\frac{t}{r_k}\right) dt. \qquad (149) \end{aligned}$$

The last integral (149) equals $r_k u_1(r/r_k)$, where u_1 is the function from the previous section (see (116)) so this last integral is *negative*. It remains to estimate from above the integral (148). Applying (146) to q and q_1, and using (141) we obtain

$$u(r) \leq \int_{[\sqrt{r}, r^3] \cap I} \log\left|1 - \frac{r^2}{t^2}\right| \left(q(t) - q_1\left(\frac{t}{r_k}\right)\right) dt + o(\omega(r)).$$

Following Kahane and Rubel, we use now the integration by parts formula

$$\int_a^b \log|1-r^2/t^2|q(t)dt = -\int_a^b \phi(t/r)tq'(t)dt + \phi(t/r)tq(t)|_a^b, \tag{150}$$

where ϕ is the function from Lemma 18, and obtain

$$u(r) \le \int_{[\sqrt{r},r^3]\cap I} \phi(t/r)t\left(|q'(t)| + \left|\frac{d}{dt}q_1(t/r_k)\right|\right)dt \tag{151}$$

$$+ \quad |\phi(t/r)|tq(t)||_{\sqrt{r}}^{r^3} + |\phi(t/r)|tq_1(t/r_k)||_{\sqrt{r}}^{r^3}. \tag{152}$$

To estimate the non-integrated terms in the last formula we use

$$\phi(t) = O(t^{-2}), \quad t \to \infty \quad \text{and} \quad \phi(t) = O(\log(1/t)), \quad t \to 0.$$

which can be obtained from the explicit expression for ϕ in Lemma 18. So

$$\left|\phi\left(\frac{t}{r}\right) tq(t)|_{\sqrt{r}}^{r^3}\right| = O(\sqrt{r}\log r) = o(\omega(r),$$

and similar estimate holds for the non-integrated term in (152) that contains q_1.

It remains to estimate the integral (151). In view of (144) and (145), this integral is at most

$$I(r) = \int_0^\infty \phi\left(\frac{t}{r}\right) tB(t)dt,$$

which is $O(\omega(r))$ by Lemma 18. Thus $u(r) \le C\omega(r)$.

To remove the C in the last inequality, we argue as follows. Let ω be the given weight. Consider another weight $\omega_1 = o(\omega)$, such that ω_1 has all properties (121), (122), (124) and (123). Perform all the above with ω_1 instead of ω. Then we obtain a function u which will satisfy $u(r) \le \omega(r)$, for r large enough.

4. Other properties of u.

Now we verify that the limit set of the subharmonic function u defined in (130) is the same $\mathcal{F}$ as in section 10.

To do this it is enough to find all limit functions of $q(tx)$ as $r \to \infty$. Let (t_n) be an arbitrary sequence tending to infinity. Let (k_n) be the sequence defined by $t_n \in [r_{k_n}, r_{k_n+1}]$. By choosing a subsequence, we may assume without loss of generality that both limits $R_0 = \lim t_n/r_{k_n} \in [1, +\infty]$ and $R_1 = \lim t_n/r_{k_n+1} \in [0, 1]$ exist. Then the following cases are possible:

Case 1. One of the limits $R_0 < +\infty$ or $R_1 > 0$. Then it follows from (141) in the definition of q that $q(t_n x) \to q_1(R_0 x)$ or $q(t_n x) \to q_1(R_1 x)$ uniformly on every compact subset of $(0, +\infty)$.

Case 2. $R_0 = \infty$ and $R_1 = 0$. Fix any number $\delta \in (0, 1)$ and let $x \in [\delta, 1/\delta]$. Then

$$\frac{d}{dx} q(t_n x) \to 0, \quad n \to \infty$$

uniformly in x, so after choosing a subsequence we obtain $q(t_n x) \to c$ uniformly with respect to x. It is easy to see that $c \in [m', m]$.

This proves that the limit set of u has the same form as in the previous section, and the rest of the arguments in that section can be repeated.

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