Oscillation of Fourier Integrals with a spectral gap

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1 Introduction

Suppose that in a real Fourier series, first m terms vanish:

$$f(x) = \sum_{n \geq m} (c_n e^{inx} + \overline{c}_n e^{-inx}), \quad f \neq 0. \tag{1}$$

Then f has at least $2m$ changes of sign on the interval $|x| \leq \pi$. For trigonometric polynomials this follows from a result of Sturm [28]; the general case is due to Hurwitz. See also problems II-141, III-184 and VI-57 in [25].

Here is a simple proof of Sturm's theorem. The number of sign changes is even. If f has at most $2(m-1)$ changes of sign then we can find a trigonometric polynomial g of degree at most $m-1$ which changes sign at the same places as f. Then fg is of constant sign which contradicts the orthogonality of f and g.

We consider the following extension of this result to Fourier integrals.

Statement 1 *Suppose that Fourier transform of a real function f is zero on an interval $(-a, a)$. Then the number of sign changes $s(r, f)$ of f on the interval $[0, r]$ satisfies*

$$\liminf_{r \to \infty} \frac{s(r, f)}{r} \geq \frac{a}{\pi}. \tag{2}$$

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We will say that a function f satisfying the assumption of Statement 1 has a *spectral gap* $(-a, a)$. In engineering literature such functions are called high-pass signals.

The proof given above permits to estimate the *upper* density of the sequence of sign changes of f, but our goal is to estimate the *lower density*.

To make Statement 1 precise, one has to define the exact meaning of the words "Fourier transform". Of the many generalizations of classical theories of Fourier transform for L^2 or for measures of bounded variation, we mention first of all the theory of temperate distributions of Schwartz [12]. A further generalization was proposed by Beurling in his lectures [4][1]. We recall the definition of Beurling's distributions. Let ω be a real function on the real line, which has the properties (3)–(5) below[2]

$$0 = \omega(0) \leq \omega(x+y) \leq \omega(x) + \omega(y), \quad x, y \in \mathbf{R}. \tag{3}$$

It follows that ω is uniformly continuous on the real line. Every even non-negative function with the property $\omega(0) = 0$, and concave on $[0, \infty)$, satisfies (3). The other two properties are

$$\int \frac{\omega(x)}{1+x^2} dx < \infty, \tag{4}$$

and

$$\omega(x) \geq \log(1 + |x|). \tag{5}$$

The space $\mathcal{S}_\omega$ of *test functions* consists of all functions ϕ in $L^1 := L^1(\mathbf{R})$, such that ϕ and its Fourier transform

$$\hat{\phi}(t) = \int \phi(x) e^{-itx} dx$$

belong to C^∞ and satisfy

$$\sup_{\mathbf{R}} |\phi^{(k)}| e^{\lambda\omega} < \infty, \tag{6}$$

$$\sup_{\mathbf{R}} |\hat{\phi}^{(k)}| e^{\lambda\omega} < \infty, \tag{7}$$

[1]Unfortunately, [4] is unpublished. There is an exposition of Beurling's theory in [7].
[2]If the limits of integration are not shown, the integral is over $\mathbf{R}$.

for all non-negative integers k and all $\lambda \geq 0$. The topology on $\mathcal{S}_\omega$ is defined by the seminorms (6) and (7). The dual space $\mathcal{S}'_\omega$ is called the space of ω-*temperate distributions*. When $\omega = \log(1+|x|)$ we obtain the space $\mathcal{S}'$ of Schwartz's temperate distributions. Fourier transform of a distribution f is defined by

$$(\hat{f}, \phi) = (f, \hat{\phi}).$$

The support of a distribution f is the complement of the maximal open set $U \subset \mathbf{R}$ such that $(f, \phi) = 0$ for all $\phi \in \mathcal{S}_\omega$ with support in U. A complex-valued locally integrable function f on the real line defines an ω-tempered distribution if it satisfies

$$\int |f(x)| e^{-\lambda\omega(x)} dx < \infty \quad \text{for some} \quad \lambda > 0. \tag{8}$$

Then

$$(f, \phi) = \int f(x)\phi(x)dx, \quad \phi \in \mathcal{S}_\omega.$$

Conditions (8) and (4) imply

$$\int \frac{\log^+ |f(x)|}{1+x^2} dx < \infty. \tag{9}$$

It is exactly this property or, more precisely, the property (4) of the weight ω which ensures that the test functions in Beurling's theory *are not* quasianalytic, and permits to construct partitions of unity in $\mathcal{S}_\omega$.

A real function $\omega \geq 1$ on the real line is called a *Beurling–Malliavin weight* (BMW) if it satisfies (4) and, in addition, has at least one of the following properties:

(i) ω is uniformly continuous, or

(ii) $\exp \omega$ is the restriction of an entire function of exponential type to the real line.

Every ω with the properties (3) and (4) is a BMW satisfying (i). BMW are important because of the following

Theorem of Beurling and Malliavin. *For every BMW ω and every $\eta > 0$ there exists an entire function g of exponential type η, such that $g \exp \omega$ is bounded on the real line.*

The references are [5, 18] and [17, Vol. 2].

Our main result is the following.

Theorem 1 *Let $\omega \geq 0$ be a BMW. If $f \neq 0$ is a real measurable function satisfying* (8) *and having a spectral gap* $(-a, a)$*, then* (2) *holds.*

In particular, this applies to locally integrable temperate distributions of the space $\mathcal{E}'$, which contains, for example, all bounded functions.

The theory of mean motion [14] suggests a stronger version of (2):

$$\liminf_{x-y\to+\infty} \frac{n(x,f)-n(y,f)}{x-y} \geq \frac{a}{\pi}. \tag{10}$$

This is not true, even for L^1 functions with bounded spectrum:

Example 1 *For every pair of positive numbers $a < b$, there exists a real entire function f of exponential type b, whose restriction to the real line is bounded and belongs to L^1, which has a spectral gap $(-a, a)$, and the property that for a sequence of intervals $[y_k, x_k]$ whose lengths tend to infinity, f has no zeros on $[y_k, x_k]$.*

Examples of functions with a spectral gap and no sign changes on *one* long interval are contained in [20].

There exist more general setting for Fourier integrals than ω-tempered distributions, namely hyperfunctions. We only consider hyperfunctions with bounded support. The name "hyperfunctions" was introduced by M. Sato in 1959, but actually Fourier transform of hyperfunctions of one variable with compact support was studied since 1920-s under the name of Borel transform [6]. Fourier integrals of hyperfunctions no longer satisfy (9): *every* entire function f of exponential type, whose indicator diagram is an interval $[ic, ib]$ on the imaginary axis, is a Fourier integral of a hyperfunction with compact support $[c, b]$. This "generalization of the Paley–Winer–Schwartz theorem to hyperfunctions" [12, v. 2, Thm. 15.1.5] coincides with Pólya's theorem, [6], [19, Ch. I, Thm. 33].

In engineering literature, an entire function of exponential type whose indicator diagram is an interval on the imaginary axis is called a *band-limited function*. We prefer to use this term because it is shorter. "Entire function of exponential type" will be abbreviated as *efet*.

Statement 1 is not true for Fourier integrals of hyperfunctions:

Example 2 *For every positive numbers $a < b$, there exists a real entire function f whose Fourier transform is a hyperfunction supported on $[-b, -a] \cup [a, b]$), and such that*

$$\liminf_{r\to\infty} s(r,f)/r < a/\pi.$$

We conclude that Statement 1 is true for distributions but not for hyperfunctions, and that condition (9) is crucial. Convergence or divergence of the integral (9) is a fundamental dichotomy in Harmonic Analysis, [4, 17]. From our point of view, the main difference between the functions which satisfy (9) and those which do not is explained by Cartwright's theorem [19, Ch. 5, Thm. 7]: condition (9) is the minimal[3] condition that implies completely regular growth in the sense of Levin and Pfluger. It is this property which is responsible for different behavior of distributions and hyperfunctions with respect to Statement 1. The situation is somewhat similar to the failure for hyperfunctions of Titchmarsh's theorem on the support of convolution [27, 15].

It is easy to construct examples of bounded band limited functions f for which the limit in (2) does not exist. However, the theory of mean motion suggests the following question: *under what additional conditions does the limit in* (2) *exist? Does it exist for exponential sums*

$$f(x) = \sum_{n=0}^{m} a_n e^{i\lambda_n x}, \quad \lambda_n \in \mathbf{R}, \quad a_j \in \mathbf{C}\ ?$$

The paper is organized as follows. In section 2 we discuss known results and conjectures about oscillation of functions with a spectral gap, in section 3 we reduce our Theorem 1 to its special case that $f \in L^1$, and construct Example 1. In section 4 we prove Theorem 1 under the additional assumption that f is real analytic and has only simple zeros on the real line. The general case is deduced in sections 5–7 by a smoothing procedure. Sections 8 and 9 are independent of the rest of the paper. In section 8 we give a brief account of Azarin's generalization of the theory of completely regular growth, which we need for construction Example 2 in Section 9.

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2 History and related results

High-pass signals are important in Electrical Engineering. Statement 1 was conjectured by Logan in his 1965 thesis [20] where he proved (2) under the

[3]It was shown by Kahane and Rubel [15] that the assumption (9) in Cartwright's theorem is essentially the weakest possible.

additional assumption that f is a band-limited function bounded on the real line. One can replace in his result the condition of boundedness on the real line by the weaker condition (9). So we have the following special case of our Theorem 1.

Proposition 1 *(Logan) Let $f \neq 0$ be a real band-limited function with a spectral gap $(-a, a)$ that satisfies (9). Then (2) holds.*

Example 2 shows that condition (9) cannot be dropped. We include a proof for three reasons: first, it is simple and gives a new proof of Sturm's theorem itself, second, we relax Logan's assumptions, and third, his thesis is not everywhere easily available.

Proof. Let b be the exponential type (bandwidth) of f, $b \geq a$. As f is real, it can be written as a sum

$$f(x) = h(x) + \overline{h}(x), \quad \text{where} \quad \overline{h}(z) := \overline{h(\overline{z})}, \tag{11}$$

and h is an band-limited function with a spectrum on $[a, b]$ which satisfies (9). For the proof of this representation (11) see Proposition 3 in the next section. Now

$$f = e^{ibx} h_1 + e^{-ibx} h_2 = \cos(bx)(h_1 + h_2) + i \sin(bx)(h_1 - h_2), \tag{12}$$

where h_1 and h_2 have their spectra on $[a - b, 0]$ and $[0, b - a]$, respectively. We conclude that $g = h_1 + h_2$ is a real efet whose indicator diagram is the interval $[i(a - b), i(b - a)]$, and g satisfies (9). Thus by Cartwright's theorem [19, Ch. V, Thm. 7], g is an efet of completely regular growth in the sense of Levin and Pfluger. In particular, the sequence of (all complex) zeros of g has a density equal to $(b - a)/\pi$. So the upper density of real zeros of g is at most $(b - a)/\pi$. On the other hand, (12) implies

$$f(n\pi/b) = (-1)^n g(n\pi/b),$$

from which it is easy to derive that $s(r, f) \geq [br/\pi] - s(r, g)$. Dividing by r and passing to the lower limit, we obtain (2). □

It is important for this proof that f is an efet. Our Theorem 1, whose proof is based on different ideas, extends Logan's result to functions whose Fourier transform has unbounded support.

The following conjecture of P.G. Grinevich is contained in [1, (1996-5)]: *"If a real Fourier integral f has a spectral gap $(-a, a)$ then the limit average density of zeros of f is at least a/π".*

In the commentary to this problem in [1], S.B. Kuksin mentioned the following result as a supporting evidence for Grinevich's conjecture. Let $\xi(t)$ be a Gaussian stationary random process, normalized by $\mathbf{E}\xi(0) = 0$ and $\mathbf{E}\xi(0)^2 = 1$, where $\mathbf{E}$ stands for the expectation. Let r be the correlation function of this process, $r(t) = \mathbf{E}\xi(0)\xi(t)$. Assume that the function r is integrable and has a spectral gap $(-a, a)$. Denote by $\mathcal{E}_T$ the random variable which is equal to the number of zeros of the random function $\xi(t)$ on $[0, T]$. Then almost surely $T^{-1}\mathcal{E}_T$ has a limit as $T \to \infty$, and this limit is at least a/π.

Other known results deal with averaged densities, like

$$S(r, f) = \int_0^r \frac{s(t, f) + s(-t, f)}{t} dt.$$

When one uses $S(r, f)$, the distinction between distributions and hyperfunctions apparently disappears. To demonstrate this, we state and prove a version of Proposition 1:

Proposition 2 *Let $f \neq 0$ be a real band-limited function with a spectral gap $(-a, a)$. Then*

$$\liminf_{r\to\infty} \frac{S(r, f)}{r} \geq \frac{2a}{\pi}. \tag{13}$$

This property is weaker than (2).

Proof. We repeat the proof of Proposition 1, but instead of using the theorem of Cartwright, apply Jensen's formula. Let $n(t, g)$ be the number of zeros of g in the disc $\{z : |z| \leq t\}$. Then, evidently, $s(t, g)+s(-t, g) \leq n(t, g)$, and Jensen's formula gives

$$\int_0^r \frac{n(t, g)}{t} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|g(re^{i\theta})| d\theta + O(1) \leq \frac{2(b - a)}{\pi}(r + o(r)), \quad r \to \infty.$$

The rest of the proof is the same as of Proposition 1. □

The following result is due to Levin, [19, Appendix II, Thm 5]. *Let $F \neq \text{const}$ be a real function of bounded variation on the real line, and dF has a spectral gap $(-a, a)$. Then*

$$\liminf_{r\to\infty} \left\{ S(r, dF) - \frac{2a}{\pi} r \right\} > -\infty. \tag{14}$$

This property neither follows from nor implies (2).

In a footnote on p. 403 of [19] Levin writes: "A similar, somewhat stronger result was obtained by M.G. Krein in the theory of continuation of Hermite-positive functions". Unfortunately, we were unable to find out what the precise formulation of Krein's result was.

Proposition 2 and Levin's theorem suggest that (13) might be true for all "Fourier integrals" of hyperfunctions having a spectral gap $(-a, a)$. Using the theorem of Beurling–Malliavin as we do in the next section permits to prove a version of Levin's theorem for functions satisfying (9), but we conjecture that (9) is not needed, that is (13) holds for arbitrary hyperfunctions with a spectral gap.

Recently Ostrovskii and Ulanovskii [23] extended and improved Levin's result as follows: *Let $dF(x)$ be a Borel measure satisfying*

$$\int \frac{|dF(x)|}{1+x^2} < \infty, \tag{15}$$

where $|dF|$ stands for the variation, and dF has a spectral gap $(-a, a)$. Then

$$\liminf_{r\to\infty} \left\{ \int_1^r \left(\frac{1}{t^2} + \frac{1}{r^2} \right) s(t, dF) dt - \frac{a}{\pi} \log r \right\} > 0,$$

and

$$\liminf_{r\to\infty} \left(S(r, dF) - \frac{2a}{\pi} r + 3 \log r \right) > 0.$$

If $dF(x) = f dx$, condition (15) is stronger than (9), but weaker than the requirement of bounded variation in Levin's theorem.

These authors [22] also proved several interesting results where the assumption about a spectral gap $(-a, a)$ is replaced by a weaker assumption that the Fourier transform of f has an analytic continuation from the interval $(-a, a)$ to a half-neighborhood of this interval in the complex plane. However, in this result they characterize the oscillation of f in terms of the Beurling–Malliavin density of sign changes, which is a sort of *upper density* rather then lower density, see, for example, [17, vol. II].

The original results of Sturm in [28] were about eigenfunctions of second order linear differential operators L on a finite interval; the case of trigonometric polynomials corresponds to $L = d^2/dx^2$ on $[-\pi, \pi]$. In 1916, Kellogg [16] gave a rigorous proof of Sturm's claim for certain class of operators, whose inverses are defined by totally positive symmetric kernels on a finite interval $[a, b]$: *Let ϕ_k be the k-th eigenfunction. Then every linear combination*

$$\sum_{k=m}^{n} c_k \phi_k \not\equiv 0, \quad n > m$$

has at least $m-1$ and at most $n-1$ sign changes on $[a, b]$.

Our paper is based on a combination of two ideas; the first is the proof of Sturm's theorem in [25, III-184], the second Sturm's own argument [28, p. 430-433] (compare [24]). We recall both proofs for the reader's convenience.

1. Write the trigonometric polynomial (1) as

$$f(x) = h(x) + \overline{h}(x), \quad \text{where} \quad h(x) = \sum_{n \geq m} c_n e^{inx},$$

then $h(x) = p(e^{itx})$ where p is a polynomial which has a root of multiplicity m at zero. By the Argument Principle, $p(z)$ makes at least m turns around zero as z describes the unit circle, so the curve $\{e^{itx} : 0 \leq x \leq 2\pi\}$ intersects the imaginary axis at least $2m$ times transversally. But $f(x) = 2\Re h(x)$ changes sign at each such intersection.

2. Use our trigonometric polynomial (1) as the initial condition of the Cauchy Problem for the heat equation on the unit circle. All coefficients will exponentially decrease with time, and the lowest order term will have the slowest rate of decrease. On the other hand, as Sturm argued, the number of sign changes of a temperature does not increase with time, [28, 24, 29]. So the number of sign changes of the initial condition is at least that of the lowest degree term in its Fourier expansion.

In sections 3-4 we develop the first idea, and in sections 5-7 the second.

To conclude this survey, we mention that Fourier Integral first appears in Fourier's work on heat propagation [11], and that study of sign changes was one of the main mathematical interests of Fourier during his whole career [10, 11].

3 Application of the theorem of Beurling and Malliavin

Given a BMW ω and $\eta > 0$, there exists an entire function g of exponential type η with the property that $g \exp \omega$ is bounded on the real line. Such function g will be called an η-multiplier. There is a lot of freedom in choosing a multiplier, so we can ensure that g has some additional properties.

First, there always exists a non-negative multiplier. Indeed, we can replace g by $g(z)\overline{g(\overline{z})}$. A non-negative multiplier g permits to reduce the proof of Theorem 1 to its special case that $f \in L^1$. Indeed, let f be a function satisfying the conditions of Theorem 1. For arbitrary $\eta \in (0, a)$ we choose a non-negative η-multiplier. Then $gf \in L^1$, has the same sequence of sign changes as f, and a spectral gap $(-a + \eta, a - \eta)$. Applying Theorem 1 to gf we obtain that the sequence of sign changes of f has lower density at least $(a - \eta)/\pi$, for every $\eta \in (0, a)$. This implies (2).

We will use this observation in sections 5-7.

Second, there always exists a multiplier whose all zeros are real. (In fact, the multiplier constructed in the original proof of the Beurling and Malliavin theorem has this property). This we will use below in the proof of Proposition 3.

Suppose that $f \in L^1$. Then Fourier transform of f is defined in the classical sense,

$$\hat{f}(t) = \int e^{-ixt} f(x) dx,$$

and $\hat{f}$ is a bounded function on the real line with the property that $f(t) = 0$ for $t \in (-a, a)$. For $0 < p < \infty$ we denote

$$\|h\|_p^* = \int \frac{|h(x)|^p}{1 + x^2} dx,$$

and define the Hardy class H^p as the set of all holomorphic functions h in the upper half-plane with the property that $\|h(. + iy)\|_p^*$ is a bounded function of y for $y > 0$.

Lemma 1 *Let f be a real function in L^1. Then there exists a function h in $H^{1/2}$ such that*

$$f(x) = h(x) + \overline{h}(x) \quad \text{a. e.,} \quad \text{and} \quad h(iy) \to 0, \quad y \to +\infty, \qquad (16)$$

where $h(x)$ is the angular limit of h. Furthermore,

$$\|h\|_{1/2}^{*} \le C_1 \|f\|_1 + C_2, \tag{17}$$

where C_1 and C_2 are absolute constants. Representation (16) *is unique.*

Proof. We define

$$h(z) = \frac{1}{2\pi} \int_0^{\infty} e^{itz} \hat{f}(t)dt, \quad \Im z > 0, \tag{18}$$

which is evidently holomorphic in the upper half-plane. Now we have for $\Im z > 0$:

$$\begin{aligned} h(z) &= \frac{1}{2\pi} \int_0^{\infty} e^{itz} \left\{ \int e^{-its} f(s)ds \right\} dt \\ &= \frac{1}{2\pi} \int f(s) \left\{ \int_0^{\infty} e^{it(z-s)} dt \right\} ds \\ &= \frac{i}{2\pi} \int f(s) \frac{ds}{z-s}. \end{aligned}$$

Taking the real part, we obtain

$$2\Re h(x+iy) = \frac{y}{\pi} \int \frac{f(s)ds}{(x-s)^2 + y^2} \, ,$$

so $2\Re h$ is the Poisson integral of f. By Cauchy–Schwarz Inequality

$$\|\Re h\|_{1/2}^{*} \le \sqrt{\pi} \|f\|_1^{1/2}. \tag{19}$$

To prove that $h \in H^{1/2}$, we use the representation of $\Im h$ as a Hilbert transform,

$$\Im h(x+iy) = \frac{1}{\pi} \int \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) \Re h(t+iy)dt,$$

and Kolmogorov's inequality,

$$m(\lambda) := \int_{|\Im h(x+iy)|>\lambda} \frac{dx}{1+x^2} \le \frac{4}{\lambda} \int \frac{|\Re h(x+iy)|}{1+x^2} dx,$$

for each $\lambda > 0$. These can be found in [17, v 1, p. 63]. We have

$$\|\Im h(.+iy)\|^*_{1/2} = \int \frac{\sqrt{|\Im h(x+iy)|}}{1+x^2} dx$$
$$= -\int_0^\infty \lambda^{1/2}\, dm(\lambda) = \frac{1}{2}\int_0^\infty \lambda^{-1/2} m(\lambda) d\lambda$$
$$\leq \pi + 2\|\Re h(.+iy)\|_1 \int_1^\infty \lambda^{-3/2} d\lambda \leq C_1\|f\|_1 + C_2.$$

Combined with (19), this implies (17).

The uniqueness statement is proved in [8, Ch. II, §2]. □

Now we want to restate the condition that $\hat{h}(t) = 0$ for $t < a$ in terms of h itself.

Lemma 2 *Let $h \in H^{1/2}$ be a function represented by Fourier integral* (18)*, where $\hat{f}$ is bounded, and $\hat{f}(t) = 0$ for $t < a$. Then h satisfies*

$$h(x+iy) = O(e^{-ay}) \quad y \to \infty. \tag{20}$$

Proof.

$$|h(x+iy)| \leq \frac{\|\hat{f}\|_\infty}{2\pi} \int_a^\infty e^{-sy} ds \leq \frac{e^{-ay}}{2\pi y}\|\hat{f}\|_\infty.$$

□

We denote by N the Nevanlinna class of functions of *bounded type* in the upper half-plane. A holomorphic function h in the upper half-plane belongs to N if h is a ratio of bounded holomorphic functions in the upper half-plane. We refer to [21, 26] for the theory of the class N. Function h from Lemma 1 belongs to N because $H^p \subset N$ for all $p > 0$. So we have the Nevanlinna representation

$$h(z) = e^{ia'z} B(z) e^{u(z)+iv(z)}, \tag{21}$$

where a' is a real number, B a Blaschke product,

$$B(z) = \prod_{n\geq 1} \left(1 - \frac{z}{z_n}\right)\left(1 - \frac{z}{\overline{z}_n}\right)^{-1}, \tag{22}$$

u the Poisson integral of $\log|h(x)|$, and v the harmonic conjugate to u. In particular,

$$J(u) := \int_{-\infty}^{\infty} \frac{|u(x)|}{1+x^2}dx < \infty. \tag{23}$$

It follows from Lemma 2 that $a' \geq a$ in (21).

We state our conclusions as

Proposition 3 *Let f be a function satisfying the conditions of Theorem 1. Then*

$$f = h + \overline{h} \quad a.\ e.,$$

where h is a function of bounded type in the upper half-plane, having representation (21) *in which $a' \geq a$. If f is an efet then h is an efet of Cartwright's class, that is*

$$\int \frac{|\log|h(x)||}{1+x^2}dx < \infty.$$

Proof. Choose $\eta \in (0, a)$. Let g be a Beurling-Malliavin multiplier of exponential type η, real on the real line and having all zeros real. Then g is of bounded type in both upper and lower half-planes, and

$$\log|g(re^{i\theta})| = \eta r \sin\theta + o(r) \quad r \to \infty, \tag{24}$$

uniformly with respect to θ for $|\theta| \in (\epsilon, \pi - \epsilon)$, for every $\epsilon > 0$. Furthermore, $gf \in L^1$ and gf has a spectral gap $(-a+\eta, a-\eta)$. According to Lemma 1,

$$gf(x) = h_1(x) + \overline{h_1}(x), \tag{25}$$

where $h_1 \in H^{1/2}$, so $h \in N$. Lemma 2 implies that

$$\log|h_1(re^{i\theta})| \leq (\eta - a) r \sin\theta + o(r) \quad r \to \infty, \tag{26}$$

uniformly with respect to θ. Dividing (25) by g (which has no zeros outside the real axis), we conclude that (16) holds with $h = h_1/g$ which evidently belongs to N. Now (24) and (26) show that

$$\log|h(re^{i\theta})| \leq -ar\sin\theta + o(r) \quad r \to \infty,$$

uniformly with respect to θ, which implies that $a' \geq a$ in (21).

If f is an efet, let b be its exponential type and F be its Fourier transform in the sense of Carleman. Then F is analytic in

$$C \backslash ([-b,-a] \cup [a,b])$$

and $F(\infty)=0$. By Laurent's theorem, $F = F_1 + F_2$, where F_1 is analytic in $C\backslash[-b,-a]$, F_2 is analytic in $C\backslash[a,b]$, and $F_j(\infty)=0,\ j=1,2$. This leads to the decomposition

$$f(x) = h^+(x) + h^-(x), \tag{27}$$

where $h^\pm$ are efet with indicator diagrams $[-ib,-ia]$ and $[ia,ib]$ respectively, so $h^\pm(iy) = O(\exp(-a|y|)),\ y \to \pm\infty$. Multiplying (27) by g, and using the uniqueness statement in Lemma 1 we obtain $h^+ = h$ and $h^- = \overline{h}$, where h is a function of the class N as above. □

Construction of Example 1. We combine Logan's method [20, Thm 5.5.1] with the theorem of Beurling and Malliavin. Without loss of generality, we may assume that $a = \pi - 2\epsilon$, and $b = \pi + 2\epsilon$, where $\epsilon > 0$. Let g_1 be a real entire function of exponential type ϵ, satisfying (9), with only simple zeros, and such that the zero set of g_1 coincides with the set of integer points on the intervals $[y_k, x_k]$:

$$g_1(n) = 0,\ g'(n) \neq 0 \quad \text{for} \quad n \in \mathbf{Z} \cap \left(\cup_{k=1}^{\infty}[y_k,x_k]\right).$$

Such function g_1 can be easily constructed if the intervals $[y_k, x_k]$ are not too long in comparison with x_k, for example, if th

$$\sum_{k=1}^{n}(x_k - y_k) \leq x_n^{\alpha} \quad \text{for some} \quad \alpha \in (0,1).$$

Let g be an entire function of type ϵ, which is positive on the real line and such that $|x|^2 g(x) g_1(x)$ is bounded for $x \in \mathbf{R}$. Such function g exists by the Beurling–Malliavin theorem (ii). Then

$$f(z) = g(z) g_1(z) \sin \pi z.$$

does not change sign on any of the intervals $[y_k, x_k]$, and $\hat{f}$ has support on

$$[-\pi - 2\epsilon, -\pi + 2\epsilon] \cup [\pi - 2\epsilon, \pi + 2\epsilon].$$

Evidently, $f \in L^1$. One can also kill multiple zeros, if desirable, by taking $f(z + 1/2) + f(z)$ instead of f. □

4 Theorem 1 for real analytic functions

To present the ideas unobscured by technical details, we prove in this section Theorem 1 for real analytic functions f whose real zeros are simple, so that the sign changes occur exactly at the zeros of f. The general case will be obtained from this special case in sections 5–7, by a smoothing procedure.

We write, as in Proposition 3,

$$f(x) = h(x) + \overline{h}(x), \tag{28}$$

as in (16), and in Proposition 3, where h has spectrum on $[a - \eta, \infty)$, and consider the Nevanlinna representation (21). Our assumptions about analyticity and simple zeros imply that v in (21) is piecewise continuous, the only jumps of $-\pi$ occur exactly at the real zeros of h (which are all simple).

Put

$$\phi(x) = \arg h(x) := a'x + \arg B(x) + v(x).$$

The Blaschke product has a continuous argument because zeros in the upper half-plane cannot accumulate to points on the real axis. Furthermore, $\arg B$ is an increasing function, which is seen by inspection of each factor of the product (22).

Let γ be the curve in the (x, y)-plane consisting of the graph of ϕ and vertical segments of length π added at the points of discontinuity of v. At each intersection of this curve with the set

$$L = \{(x, y) : x \in \mathbf{R},\ y - \pi/2 \in \pi\mathbf{Z}\}, \tag{29}$$

the number $h(x)$ is purely imaginary, that is $f(x) = 0$ by (28).

So we want to estimate from below the number of intersections of γ with L over the intervals $[0, r]$.

We fix $\epsilon \in (0, 1/2)$ and prove that

$$\phi(x) \geq a'x + v(x) > (a' - \epsilon)x, \tag{30}$$

for all x, except a set of zero density.

It will immediately follow from (30) that the number of intersections $\gamma \cap L$ has lower density at least a'/π. So it remains to prove (30).

We recall that v is harmonically conjugate to u, and that u satisfies (23). According to Kolmogorov's inequality [17, v 1, p. 63]

$$\int_{|v(x)|>\lambda} \frac{dx}{1 + x^2} \leq \frac{4}{\lambda} \int_{-\infty}^{\infty} \frac{|u(x)|}{1 + x^2} dx,$$

for each $\lambda > 0$.

We break u into two parts with disjoint supports, $u = u_0 + u_1$, where the support of u_0 belongs to $[-r_0, r_0]$ for some $r_0 > 0$ and u_1 satisfies

$$\int \frac{|u_1(x)|}{1+x^2} dx = \int_{|x|>r_0} \frac{|u_1(x)|}{1+x^2} dx < \epsilon^2/8, \tag{31}$$

which is possible in view of (23). Let $v_j = \mathcal{H} u_j,\ j = 0, 1$; where $\mathcal{H}$ stands for the Hilbert transform,

$$\mathcal{H} u(x) = \lim_{y \to 0+} \frac{1}{\pi} \left(\int \frac{x-t}{(x-t)^2 + y^2} + \frac{t}{t^2+1} \right) u(t)\, dt.$$

Lemma 3 $|v_0(x)| \leq J(2r_0 + r_0^{-1})/\pi$ *for* $|x| > 2r_0$, *where* $J = J(u)$ *is defined in* (23).

Proof.

$$\begin{aligned}
|v_0(x)| &\leq \frac{1}{\pi} \left| \int_{-r_0}^{r_0} \frac{u_0(t)}{x-t} dt \right| + \frac{1}{\pi} \left| \int_{-r_0}^{r_0} \frac{t u_0(t)}{t^2+1} dt \right| \\
&\leq \frac{1}{\pi} r_0^{-1} \int_{-r_0}^{r_0} |u_0(t)| dt + \frac{1}{\pi} r_0 J \\
&\leq \frac{1}{\pi} r_0^{-1} (1 + r_0^2) J + \frac{1}{\pi} r_0 J \\
&= J(2r_0 + r_0^{-1})/\pi.
\end{aligned}$$

□

Now we prove that for every $x > 2$ there exists

$$x' \in [(1-\epsilon)x, x],$$

such that

$$v_1(x') > -\epsilon x. \tag{32}$$

Suppose that this is not so. Then we apply Kolmogorov's inequality to v_1 and u_1 with $\lambda = \epsilon x$, and (31):

$$\int_{(1-\epsilon)x}^{x} \frac{dt}{2t^2} < \int_{(1-\epsilon)x}^{x} \frac{dt}{1+t^2} < \frac{4}{\epsilon x} \int \frac{|u_1(x)|}{1+x^2} dx < \frac{\epsilon}{2x}.$$

Evaluating the integral on the left we conclude $\epsilon/(1-\epsilon) < \epsilon$, a contradiction. This proves (our special case of) Theorem 1.

We state a more quantitative version of the result we just proved:

Proposition 4 *Let f be a function satisfying the conditions of Theorem 1. Suppose that f is real analytic and has only simple zeros on the real line. Write $f = h + \overline{h}$ as in (28), and let h be represented by the formula (21), with $J = J(u)$ as in (23). Suppose that*

$$\int_{|x|>r_0} \frac{|\log|h(x)||}{1+x^2} < \epsilon^2/8$$

for some $r_0 > 1$ and $\epsilon \in (0, 1/2)$. Then

$$s(r, f) \geq (a - \epsilon)r/\pi - J(2r_0 + r_0^{-1})/\pi - 1 \quad \textit{for} \quad r > 2r_0.$$

□

5 Heating

In this and the next two sections we assume that $f \in L^1$ in Theorem 1. This does not restrict generality, as was explained in the beginning of section 4.

If f is not real analytic, or has multiple zeros on the real line, we "heat" it[4]. This means that we replace our f by the convolution[5] with the heat kernel,

$$f_t = K_t * f, \quad f_0 = f, \tag{33}$$

$$K_t(x) = \frac{1}{\sqrt{\pi t}} e^{-x^2/t}.$$

Evidently, f_t are real analytic with respect to x for all $t > 0$. All

$$\hat{f}_t = \hat{K}_t \hat{f} = \exp(-s^2 t)\hat{f}$$

have the same support because $\hat{K}_t$ never vanishes.

Pólya [24, 29] proved that f_t has at most as many sign changes on the real line as f does. (This assertion was stated by Sturm for the case of finite interval). However, we cannot use this result[6] because our functions

[4]using an expression of Petermichl and Volberg.

[5]In the works on heat equation this is called a Poisson integral. We *don't do this* to avoid confusion with the harmonic Poisson integral.

[6]Probably it is possible to derive what we need from Pólya's result. However we think it is useful to give an independent proof of this generalization of Sturm–Pólya's theorem.

have infinitely many sign changes, and we have to control their number on every interval $[0, r]$. So we will prove the necessary generalization of Pólya's theorem.

Our approach is closer to the original approach of Sturm rather than that of Pólya.

In this section we show that heating does not destroy the conditions of Proposition 4, and in the next two sections we deal with the behavior of sign changes under heating, and also with multiple roots which f may have on the real line.

Lemma 4 *Let $f \in L^1$ be a real function with a spectral gap $(-a, a)$, and $f_t = K_t * f$. Define h_t by (18) using f_t instead of f.*

Then $\|h_t\|_1 \leq \|h\|_1$, and $J(\log|h_t|) \leq C_1$, where C_1 is independent of t. Further, for every $\epsilon > 0$ there exist $r_0 > 0$ and $t_0 > 0$, such that for all $t \in (0, t_0)$ we have

$$\int_{|x|\geq r_0} \frac{|\log|h_t(x)||}{1+x^2} dt < \epsilon. \tag{34}$$

We emphasize that r_0 and C_1 are independent on t. They only depend of h and ϵ.

Proof. First, of all,

$$\int |f_t(x)|dx = \int |K_t * f|(x)dx \leq \int (K_t * |f|)(x)dx = \int |f(x)|dx,$$

so $\|f_t\|_1 \leq \|f\|_1$. Using Lemma 1 we obtain $\|h_t\|^*_{1/2} \leq C$, with C independent of t. Thus

$$\sqrt{|h_t(x)|} = k_t(x)(1+x^2), \quad \text{where} \quad \|k_t\|_1 \leq C. \tag{35}$$

We have

$$\begin{aligned} &\log^+|h_t| \leq 2\log^+|k_t| + 2\log(1+x^2) \\ &\leq 4\left(\sqrt{|k_t|} - 1\right)^+ + 2\log(1+x^2). \end{aligned} \tag{36}$$

Let $u_t(x) = \log|h_t(x)|$ for real x and $t \geq 0$. Dividing (36) by $1+x^2$, integrating and using (35) gives

$$J(u_t^+) = \int \frac{u_t^+(x)}{1+x^2} dx < C, \tag{37}$$

where C is independent of t. Similarly we obtain from (36) that

$$\int_{|x|\geq r_0} \frac{u_t^+(x)}{1+x^2}dx < \epsilon,$$

with r_0 independent of t.

Property (37) makes possible to extend u_t^+ to the upper half-plane by Poisson's formula. We continue to denote the extended function by u_t^+. Notice that $h_t \in N$ for all t, and $u_t^+(x+iy) - ay$ is a positive harmonic majorant of $\log |h_t|$ in the upper half-plane.

Now we prove

$$J(u_t^-) < C, \tag{38}$$

with C independent of t, and (34) for the negative part of u_t. Fix a point z_0 in the upper half-plane, such that $\delta = |h(z_0)| > 0$. As $h_t(z_0) \to h(z_0)$ as $t \to 0$, we conclude that $h_t(z_0) > \delta/e$ when t is small enough. Let b be the *true left end* of the support of $\hat{h}_t$. It is important to notice that b is *independent* of t, because $\hat{h}_t = \hat{K}_t \hat{h}$. Then

$$u_t(z_0) - b\Im z_0 \geq \log\delta - 1 > -\infty, \tag{39}$$

when t is small enough. This implies (38). It remains to prove (34) for the negative part of u_t. For psychological reasons it is better to work in the unit disc $\mathbf{U}$ instead of the upper half-plane. The fractional-linear transformation $T(z) = (z-i)/(z+i)$ maps the upper half-plane onto $\mathbf{U}$, $T(\infty) = 1$, and we put $\zeta_0 = T(z_0)$, and

$$w_t = u_t \circ T^{-1} - b\Im T^{-1}. \tag{40}$$

As a consequence of (39) we have

$$w_t(\zeta_0) \geq \log\delta - 1 > -\infty. \tag{41}$$

The measure $dx/(1+x^2)$ on the real line corresponds to the measure $d\theta$ on the unit circle $\mathbf{T} = \{e^{i\theta} : \theta \in \mathbf{R}\}$.

It follows from (37) that each w_t is a difference of positive harmonic functions in the unit disc, so it is the Poisson integral of some charge μ_t of bounded variation on the unit circle. The constant b in (40) comes from the Nevanlinna representation of h_t similar to (21), and this constant does not depend on t. So all charges μ_t have an atom of mass exactly $-b$ at the point 1.

Let $\mu_t = \mu_t^+ - \mu_t^-$ be the Jordan decompositions. Conditions (41) and (37) imply that μ_t are of bounded total variation, with a bound independent of t. So we have weak convergence $\mu_t \to \mu_0,\ t \to 0$. Let ϕ be a continuous real valued function on the unit circle, which is identically equal to 1 in some neighborhood of the point 1, and at the same time

$$\left| b - \int_{\mathbf{T}} \phi |\mu_0| \right| < \epsilon/2,$$

where $|\mu_0| = \mu_0^+ + \mu_0^-$ is the variation of μ_0. Then there exists t_0 such that

$$\left| b - \int_{\mathbf{T}} \phi |\mu_t| \right| < \epsilon, \quad \text{for} \quad 0 \leq t \leq t_0. \tag{42}$$

When translated back to the real line from the unit circle, this gives (34). □

6 Preliminaries on temperatures

Here we collect for the reader's convenience some facts about convolutions (33) of real functions with the heat kernel. We use the convenient notation[7]

$$u(x,t) = f_t(x) \tag{43}$$

and consider u in the upper half-plane $\{s = (x,t) : t \geq 0, x \in \mathbf{R}\}$.

The function u in (43) is a solution of the heat equation in the open upper half-plane:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{44}$$

Such functions are called *temperatures*. Formula (33) solves the initial value problem on an infinite rod (the x-axis) with given initial temperature $f(x)$. A standard reference on the subject is [9]. Here is the precise statement about the boundary behavior of u which is a slight generalization of [9, 1.XVI.7]:

[7] We apologize for such abuse of the letter u, but the harmonic function u of sections 3-5 will not appear anymore until the end of section 7.

Lemma 5 *Let f be a real function from L^1. Then for every $x \in \mathbf{R}$,*

$$\liminf_{t\to 0} u(x,t) \geq \liminf_{\epsilon\to 0+} \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(t)dt.$$

This is a general property of positive symmetric kernels. Radial limits can be replaced by non-tangential limits, and even by limits from within parabolas tangent to the real line at x. It follows that at every Lebesgue density point x of f, the limit $\lim_{t\to 0+} u(x,t)$ exists and equals $f(x)$.

Next lemma (due to L. Nirenberg) is called the Strong Minimum Principle [9, 1.XV.5]

Lemma 6 *Let D be a bounded region in the horizontal strip $P = \{s = (x,t) : 0 < t < T\}$, and u a temperature in D. Suppose that*

$$\liminf_{s\to\sigma} u(s) \geq 0, \quad \textit{for all} \quad \sigma \in \partial D \cap (P \cup (\mathbf{R}\times\{0\})). \tag{45}$$

Then $u \geq 0$ in D, and if $u(s) = 0$ for some point $s \in D$ then $u \equiv 0$ in D.

We need an extension of the Minimum Principle, analogous to the Phragmén–Lindelöf Theorem in the theory of harmonic functions:

Lemma 7 *Let D be a region as in Lemma 6, and u a temperature in D. Suppose that u is bounded from below, and (45) holds for all but finitely many points $\sigma \in \partial D \cap (\mathbf{R}\times\{0\})$, and the finite set of exceptional points belongs to the real axis. Then the same conclusion as in Lemma 6 holds.*

Proof. Let $x_1, x_2, \ldots, x_n$ be the exceptional points on the real axis. Consider the auxiliary function

$$w(s) = \begin{cases} \displaystyle\sum_{k=1}^{n} \log^+ \frac{1}{|s-x_k|}, & s \in \mathbf{R}\times\{0\}, \\ \\ (K_t * w(.,0))(x), & s = (t,x),\ x \in \mathbf{R}, t > 0. \end{cases}$$

Then w is a positive temperature in P, and

$$w(s) \to +\infty \quad \text{as} \quad s \to x_k,\ \ s \in P,\ \ 1 \leq k \leq n.$$

So, for every $\epsilon > 0$, the function

$$u_\epsilon = u + \epsilon w$$

satisfies all conditions of Lemma 6. So $u_\epsilon \geq 0$, that is $u(z) \geq -\epsilon w(z)$. Letting $\epsilon \to 0$, we conclude that $u \geq 0$. So u satisfies the conditions of Lemma 6, and the conclusions of Lemma 6 hold for u. □

Lemma 8 *Let u be a temperature in some region D of (x,t)-plane. Then multiple zeros of the functions $x \mapsto u(x,t)$ are isolated in D.*

Proof. Suppose that u has a non-isolated multiple zero, Let $m \geq 2$ be the minimal multiplicity of such zero. Then there exists an analytic germ $g(t)$ which gives the position of such multiple zero for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some t_0 and $\epsilon > 0$. So we have

$$u(x,t) = (x - g(t))^m v(x,t),$$

in a neighborhood of $(g(t_0), t_0)$. Here v is a real analytic function

$$v(g(t_0), t_0) \neq 0.$$

We differentiate, and see that the lowest order term in $\partial^2 u / \partial x^2$ is

$$m(m-1)(x - g(t))^{m-2} v(x,t),$$

while all terms in $\partial u / \partial t$ are of order at least $m - 1$. So u cannot satisfy the heat equation. □

7 Heating, Part II

In this section we complete the proof of Theorem 1. Let $f \neq 0$ be a real integrable function, such that its Fourier transform $\hat{f}$ has a gap $(-a, a)$. Choose an arbitrary $\epsilon > 0$. Let r_0 be the number defined in Lemma 4. We will estimate the number of sign changes of f on the interval $[0, r]$, where $r > 2r_0$. If f has infinitely many sign changes on $[0, r]$ then there is nothing to prove. So we assume that the number of sign changes is finite on $[0, r]$. A *zero place* of f is defined as a maximal closed interval I, (which may degenerate to a point) such that $f|_I = 0$ a.e. A zero place $I = [c, d]$ is called a *place of sign change* of f if $(x - c)f(x)$ has constant sign in a neighborhood of I. The complement of the union of the places of sign changes consists of open intervals which are called *intervals of constancy of sign*. We write $I_1 < I_2$ to mean that the intervals I_1 and I_2 are disjoint and I_2 is on the right of I_1.

Let $0 < I_1 < I_2 < \ldots < I_n < r$ be the places of sign changes. We assume that

$$n \geq 2, \ f(0) < 0, \ f(r) < 0, \tag{46}$$

and that 0 and r are Lebesgue density points of f. This assumption does not restrict generality.

Let $f_t = K_t * f$, and let t_0 be the number from Lemma 4. We are going to show, that for t small enough, the number of sign changes of f_t on $[0, r]$ does not exceed that of f for $t \in (0, t_0)$.

Using Lemma 5 and negativity of f at its Lebesgue points 0 and r, we achieve that

$$\inf\{f_t(x) : x \in \{0, r\}, 0 < t < T\} < 0, \tag{47}$$

by choosing $T \in (0, t_0)$ small enough. We recall that f_T is real analytic. Using Lemma 8 we ensure that f_T has only simple zeros.

We are going to prove that

$$s(r, f_T) \leq s(r, f). \tag{48}$$

Assume first that f is bounded in some neighborhood of the union $\cup_{k=1}^{n} I_k$. As f_T is real analytic, every place of sign change of f_T is one point. We consider a maximal interval $\ell = (y_1, y_2) \subset [0, r]$ of sign constancy of f_T, where f_T is non-negative, but y_1 and y_2 are the places of sign changes of f_T. Define the strip $P = \{s = (x, t) : 0 < t < T\}$ as in Lemmas 6 and 7. Denote, as in (43), $u(s) = u(x, t) = f_t(x)$. Let D be the connected component of the set

$$\{s \in P : u(s) > 0\}, \quad \text{such that} \quad \partial D \supset \ell.$$

Notice that $u(s) = 0$ for $s \in \partial D \cap P$. Then D is bounded because it is contained in the rectangle

$$\{(s, t) : 0 < x < r, \, 0 < t < T\}$$

in view of (47). We claim that

$$\partial D \cap \{(x, t) : t = T\} = \overline{\ell} = [y_1, y_2] \times \{T\}. \tag{49}$$

Indeed, on those two intervals ℓ^- and ℓ^+ of constant sign which are adjacent to ℓ, the sign is negative, so these two intervals cannot intersect ∂D. If there is an interval, say ℓ^*, on the line $t = T$, which belongs to ∂D, and $\ell^* \cap \ell = \emptyset$, we suppose, for example that ℓ^* is on the same side of ℓ as ℓ^+. But then the component D^+ of the set $\{s \in P : u(s) < 0\}$ which has ℓ^+ on the boundary has closure in the upper half-plane (being separated by D from the x-axis), and this contradicts Lemma 6. This proves our claim (49).

If $\partial D \backslash \ell \subset P$, then $u \equiv 0$ in D by Lemma 6, and thus $u \equiv 0$ in the upper half-plane because u is real analytic. If

$$\partial D \backslash \ell \subset P \cup (I_1 \cup \ldots \cup I_n) \times \{0\},$$

we arrive at a contradiction in the similar way using Lemma 7 instead of Lemma 6. Here we used our temporary assumption that f was bounded in a neighborhood of $I_1 \cup \ldots \cup I_n$.

The conclusion is that ∂D intersects one of the intervals J, a component of the complement

$$[0, r] \backslash \cup_{k=1}^{n} I_k.$$

But then ∂D contains this interval J completely. This is because there is a neighborhood U of J such that $u(s) > 0$ for $s \in U \cap P$, which follows from Lemma 5 combined with Lemma 6. Evidently, this U cannot intersect $\partial D \cap P$.

Thus ∂D contains exactly one interval ℓ of sign constancy of f_T and at least one interval of sign constancy of f. As different regions D are evidently disjoint, we conclude that f has at least as many changes of signs on $[0, r]$ as f_T. This proves (48).

It remains to get rid of the additional assumption that f is bounded in a neighborhood of $I_1, \ldots, I_n$. Let U be a compact neighborhood of these intervals in $\mathbf{R}$, such that $U \cap \{0, r\} = \emptyset$. For every positive integer N we define

$$f^N(x) = \begin{cases} f(x) & \text{for} \quad x \notin U, \\ f(x) & \text{if} \quad |f(x)| \leq N, \\ N & \text{if} \quad f(x) > N, \ x \in U \\ -N & \text{if} \quad f(x) < -N, \ x \in U. \end{cases}$$

If N is large enough, f^N has the same number of sign changes on $[0, r]$ as f. Furthermore, $f^N \to f$ in L^1 as $N \to \infty$, because $f(x) = f^N(x)$ for $x \notin U$. As U is disjoint from the set $\{0, r\}$, the convergence $K_t * f^N \to f_t$ is uniform on $\{0, r\} \times [0, T]$, so for N large enough our functions $K_t * f^N$ are all strictly negative on this set. So the previous proof applies to $K_T * f^N$, and we conclude that $s(r, K_T * f^N) \le s(r, f)$. It remains to apply the observation that $s(r, K_T * f^N) \to f_T$ pointwise, and thus $s(r, f_T) \le s(r, f)$. So we proved (48) in full generality.

Completion of the proof of Theorem 1. It remains to put the pieces together. Let f be a function satisfying the conditions of Theorem 1. Assume wlog that 0 is a Lebesgue point of f and that $f(0) < 0$. Suppose, by contradiction, that for some $\eta \in (0, a/3)$ we have

$$\liminf_{x\to\infty} \frac{s(x, f)}{x} < \frac{a - 3\eta}{\pi},$$

and let $x_k \to \infty$ be a sequence for which

$$s(x_k, f) < (a - 3\eta)x_k/\pi. \tag{50}$$

Apply the theorem of Beurling and Malliavin to find a multiplier g of type η, such that $g(x) \ge 0$ for $x \in \mathbf{R}$. Then $gf \in L^1$ has the same sequence of sign changes as f, and a spectral gap $(-a+\eta, a-\eta)$. We may assume that x_k are Lebsegue density points with $gf(x_k) < 0$. For $t > 0$, let $(fg)_t = K_t * (fg)$ and let

$$(fg)_t = h_t + \overline{h_t}$$

be the decomposition which exists by Lemma 1. Using Lemma 4, find $r_0 > 0$ and $t_0 > 0$ such that (34) holds with $\epsilon = \eta^2/8$. Choose $r = x_k > 2r_0$ so that (50) is satisfied, and

$$(a - 2\eta)r/\pi - C_1(2r_0 + r_0^{-1})/\pi - 1 > (a - 3\eta)r/\pi, \tag{51}$$

where C_1 is the upper bound for $J(\log |h_t|)$ from Lemma 4. Then choose $t \in (0, t_0)$ so that $(fg)_t$ has only simple zeros (Lemma 8) on the real line (Lemma 8), and the number of these zeros on the interval $(0, r)$ is at most $s(r, f) = s(r, gf)$, which is guaranteed by (48). Now by Proposition 4, applied to $(gf)_t$, and (51) we have

$$s(r, f) = s(r, gf) \ge s(r, (gf)_t) > (a - 3\eta)r/\pi,$$

where $r = x_k$, which contradicts (50). This proves the theorem. □

8 Limit sets of entire functions

Cartwright's theorem mentioned in sections 1 and 2 indicates that constructing an example of an efet whose indicator diagram is an interval of the imaginary axis, and which does not have completely regular growth, may be a nontrivial task. First such examples were constructed by Redheffer, Roumieu [27] and by Kahane and Rubel [15]. Their purpose was to show that Titchmarsh's theorem on the support of convolution fails for hyperfunctions with compact support. However, all these examples are still too regular for our purposes, and we need the theory of limit sets, which generalizes the theory of completely regular growth. It is due to Azarin, Giner [2, 3], Hörmander and Sigurdsson [13]. Here we collect the necessary facts from this theory.

Let U^* be the set of all subharmonic functions in the plane satisfying

$$\limsup_{|z|\to\infty} |z|^{-1} u(z) < \infty,$$

with induced topology from the space of Schwartz distributions $\mathcal{D}'(\mathbf{C})$, and

$$U(\sigma) = \{u \in U^* : u(0) = 0,\ \sup_{z\in\mathbf{C}} |z|^{-1} u(z) \leq \sigma,\ \},$$

for $\sigma > 0$. We recall that $\mathcal{D}'(\mathbf{C})$ is a metric space. We denote $U = \cup_{\sigma>0} U(\sigma)$.

A one-parametric group A of operators

$$(A_t u)(z) = t^{-1} u(tz), \quad t > 0,$$

acts on U^*. The sets $U(\sigma)$ are A-invariant.

For a function $u \in U^*$ we define the *limit set* $\mathrm{Fr}\,[u] = \mathrm{Fr}_\infty[u]$ as the set of all limits

$$\lim_{n\to\infty} A_{t_n} u \quad \text{for} \quad t_n \to \infty.$$

Similarly, $\mathrm{Fr}_0[u]$ is defined, using sequences $t_n \to 0$. Each limit set $\mathrm{Fr}_\infty[u]$ or $\mathrm{Fr}_0[u]$ is a closed connected A-invariant subset of $U(\sigma)$ for some $\sigma > 0$. If f is an efet then $\log|f| \in U^*$ and we define the *limit set of f* as $\mathrm{Fr}\,[\log|f|]$. For every limit set $\mathrm{Fr}\,[u]$, the function

$$v(z) = \sup\{w(z) : w \in \mathrm{Fr}\,[u]\}, \tag{52}$$

is A-invariant and subharmonic. All such functions have the form

$$v(re^{i\theta}) = rh(\theta), \quad \text{where} \quad h'' + h \geq 0, \tag{53}$$

that is $h'' + h$ is a non-negative measure. Functions h with this property are called *trigonometrically convex*. Function h defined by (52) and (53) is called the *indicator* of u. If f is an efet, and h the indicator of $\log|f|$ then h coincides with the classical Phragmén–Lindelöf indicator of f. The *indicator diagram* is the closed convex set in the plane whose support function is h.

Different criteria for a subset $\mathcal{F} \subset U$ to be a limit set were found in [3] and [13]. The following result is from [3].

Proposition 5 *Fix $\sigma > 0$. For a closed connected A-invariant subset $\mathcal{F} \subset U(\sigma)$, the following conditions are equivalent:*
a) $\mathcal{F} = \mathrm{Fr}\,[u]$ for some $u \in U^*$,
b) $\mathcal{F} = \mathrm{Fr}\,[\log|f|]$ for some efet f, and
c) There exists a piecewise-continuous map

$$\mathbf{R}_{>0} \to U(\sigma), \quad t \mapsto v_t$$

with the properties

$$\mathrm{dist}\,(A_\tau v_t,\, v_{\tau t}) \to 0, \quad t \to \infty,$$

and

$$\mathrm{clos}\,\{v_t : t \in (t_0, \infty)\} = \mathcal{F}, \quad \forall t_0 > 0.$$

Here are some simple examples of sufficient conditions derived from Proposition 5.

1. One-point limit set. This characterizes completely regular growth in the sense of Levin–Pfluger.

2. One periodic orbit. Let u be a subharmonic function with the property that $A_T u = u$ for some $T \neq 1$. Then

$$\{A_t u : 1 \leq t \leq T\}$$

is a limit set. One can show that in this case the indicator diagram cannot be a non-degenerate interval of the imaginary axis, so this type of functions is not appropriate for out purposes.
2. The closure of a single orbit,

$$\{A_t u : 0 < t < \infty\} \cup \mathrm{Fr}_0[u] \cup \mathrm{Fr}_\infty[u], \quad \text{where} \quad u \in U(\sigma)$$

is a limit set if and only if

$$\mathrm{Fr}_0[u] \cap \mathrm{Fr}_\infty[u] \neq \emptyset.$$

Again, in this case the indicator diagram cannot be a non-degenerate interval of the imaginary axis.

3. "An interval". If u_0 and u_1 are two A-invariant functions in U then the set

$$\{tu_0 + (1-t)u_1 : 0 \leq t \leq 1\}$$

is a limit set. Examples in [15] are of this sort. The efet constructed in [15] has indicator diagram $[-ib, ib]$ and the *lower* density of zeros is strictly less than b/π. We need an example of efet with the indicator diagram $[-ib, ib]$ and the *upper* density of zeros strictly greater that b/π.

We combine the last two examples.

Lemma 9 *Let u be a function in U with the properties*

$$\mathrm{Fr}_0[u] = \{u_0\} \quad \textit{and} \quad \mathrm{Fr}_\infty[u] = \{u_1\}.$$

Then

$$\{A_t u : 0 < t < \infty\} \cup \{tu_0 + (1-t)u_1 : 0 \leq t \leq 1\}$$

is a limit set.

This easily follows from the general criterion in Proposition 5.

Now we describe the relation between the limit set and the distribution of zeros of an efet. Consider the set of all Borel measures in $\mathbf{C}$ (non-negative and such that the measure of every compact set is finite). The analog of operators A_t for measures is

$$(B_t\mu)(E) = t^{-1}\mu(tE), \quad \text{for Borel sets} \quad E \subset \mathbf{C}.$$

Laplace operator $(2\pi)^{-1}\Delta$ splits A_t and B_t:

$$\Delta A_t = B_t \Delta. \tag{54}$$

We denote by V^* the set of all measures μ, which satisfy

$$\limsup_{r\to\infty} r^{-1}\mu(D(r)) < \infty,$$

where $D(r) = \{z \in \mathbf{C} : |z| \leq r\},\ r \geq 0$. We also define the subsets

$$V(\sigma) = \{\mu \in V^* : r^{-1}\mu(D(r)) \leq \sigma,\ 0 < r < \infty\}, \quad \sigma > 0,$$

and $V = \cup_{\sigma>0} V(\sigma)$. Laplace operator is continuous in U and sends U to V (however, this map is not surjective, and the image of $U(\sigma)$ is not equal to $V(\sigma')$ for any $\sigma > 0$). Given a measure $\mu \in V^*$, we define the limit set $\mathrm{Fr}\,[\mu]$ as the set of all limits

$$\lim_{n\to\infty} B_{t_n}\mu \quad \text{for} \quad t_n \to \infty.$$

It follows from (54) that for every $u \in U^*$ we have

$$(2\pi)^{-1}\Delta\,(\mathrm{Fr}\,[u]) = \mathrm{Fr}\,\left[(2\pi)^{-1}\Delta u\right].$$

If f is entire then $(2\pi)^{-1}\Delta \log|f|$ is the counting measure of zeros of f. So the asymptotic distribution of zeros is reflected in the Riesz measures of the elements of the limit set. Let us make this more precise. Two measures in U^* are called equivalent if

$$B_t(\mu_1 - \mu_2) \to 0 \quad \text{as} \quad t \to \infty.$$

This implies $\mathrm{Fr}\,[\mu_1] = \mathrm{Fr}\,[\mu_2]$. Let $T : \mathbf{C} \to \mathbf{C}$ be a map with the property

$$T(z) - z = o(z), \quad z \to \infty. \tag{55}$$

We recall that push-forward $T_*\mu$ of a measure by T is defined by $(T_*\mu)(E) = \mu(T^{-1}(E))$. If $\mu \in V^*$, and a map T satisfies (55), then $T_*\mu$ is equivalent to μ. For each $\mu \in V^*$ one can construct a map T with the property (55) such that $T_*\mu$ is a counting measure of a divisor in $\mathbf{C}$. This explains the implication b)→a) in Proposition 5.

Lemma 10 *Let μ be a measure in V^*. Suppose that all measures in* $\mathrm{Fr}\,[\mu]$ *are supported on the real line and have the form $d(x)dx$ where $d(x) \leq 1$. Then there exists a measure μ', which is equivalent to μ and which is supported on the integers, and $\mu'(n) \in \{0, 1\}$ for each integer n.*

Proof. First we project our measure μ on the real line by the map

$$T(re^{i\theta}) = \begin{cases} r, & |\theta| < \pi/2, \\ -r, & |\theta - \pi| \leq \pi/2. \end{cases}$$

This map does not satisfy (55) but it is easy to see that $\mu'' = T_*\mu \sim \mu$.

Second, let F be the distribution function of μ'', that is $\mu'' = dF$ and $F(0) = 0$. Then we set $F_1(x) = F([x] + 1)$, where [.] stands for the integer part, and put $\mu' = dF_1$. □

9 Example of a hyperfunction

Here we construct Example 2 assuming, without loss of generality, that $a+b = 2\pi$. We begin with a smooth negative function $u = ku_0$ with support on $[0, 2]$, for example, we can take

$$u(x+1) = \begin{cases} -k(1-x^2)^2, & |x| \leq 1, \\ 0, & |x| > 1, \end{cases}$$

where $k > 0$ is a parameter to be specified later. Then we extend u to $\mathbf{C}\backslash\mathbf{R}$ by Poisson's integral. The resulting function u is a delta-subharmonic function in $\mathbf{C}$, whose Riesz charge is supported on $\mathbf{R}$ and has the form $dQ(x) = q(x)dx$, where q is a smooth function. So we have,

$$u(z) = \int \log|z-t|dQ(t) = \int \log|z-t|q(t)dt.$$

For the function u as above we can explicitly compute Q and q:

$$Q(x+1) = k(x^2-1)^2 \log\left|\frac{x+1}{x-1}\right| - 2kx^3 + \frac{10}{3}kx,$$

and

$$q(x+1) = 4kx(x^2-1)\log\left|\frac{x+1}{x-1}\right| - 8kx^2 + \frac{16}{3}k.$$

We put

$$-m = \min_{x\in\mathbf{R}} q(x) < 0, \tag{56}$$

and

$$\eta = \max_{x\geq 0} \frac{Q(x)}{x} > 0. \tag{57}$$

Now we choose and fix k, small enough, so that

$$m + \max_{\mathbf{R}} q < 1. \tag{58}$$

We define

$$Q_1(x) = Q(x) + mx, \quad \text{so that} \quad q_1 = Q_1' = q + m \geq 0. \tag{59}$$

in view of (56), and thus the function

$$u_1(z) = \int \left(\log|z-t| + \Re\left(\frac{z}{t}\right)\right) dQ_1(t) = u(z) + \pi m|\Im(z)|, \quad z \in \mathbf{C}, \quad (60)$$

is subharmonic in $\mathbf{C}$ and belongs to the class U defined in the previous section. We have

$$\mathrm{Fr}_\infty[u_1] = \{\pi m|\Im(.)|\} \quad \text{and} \quad \mathrm{Fr}_0[u_1] = \{\pi m'|\Im(.)|\}, \quad (61)$$

where $\Im(.)$ is the function $z \mapsto \Im(z)$, and

$$m' = m + q(0) < m.$$

Now by Lemma 9, the set

$$\mathcal{F} := \{A_t u_1 : t \in \mathbf{R}\} \cup \{t|\Im(.)| : \pi m' \leq t \leq \pi m\} \subset U$$

is a limit set of an efet. Evidently,

$$\sup\{w(z) : w \in \mathcal{F}\} = \pi m|\Im(z)|. \quad (62)$$

Let g be an entire function of exponential type m, such that

$$\mathrm{Fr}\,[\log f] = \mathcal{F}.$$

Then in view of (62), the indicator diagram of g is the interval $[-\pi m i, \pi m i]$. In other words, g is a hyperfunction whose Fourier transform has support on $[-\pi m, \pi m]$.

In addition, we require that all zeros of g, except $o(r)$ of them be simple and located at integers, which is possible by Lemma 10 because the Riesz measures of all elements of $\mathcal{F}$ are concentrated on the real line, and their densities do not exceed 1 in view of (58) The upper density of zeros of g is

$$\max_{x \geq 0} Q_1(x)/x = m + \eta. \quad (63)$$

Finally we set

$$f(z) = g(z) \sin \pi z.$$

Then, according to [12, v. 2, Thm. 15.1.5], f is a hyperfunction whose Fourier transform is supported on

$$\pi[-1-m,\, -1+m] \cup \pi[1-m,\, 1+m],$$

while the sign changes occur only at those integers which are not zeros of g, that is the lower density of sign changes is at most $1 - m - \eta < 1 - m$ in view of (63). □

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