PROJECT SUMMARY

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Geometric function theory studies relations between properties of a meromorphic function and geometry of its image. In the case of univalent functions, the image is a domain on the Riemann sphere. In the general case, it is a "Riemann surface spread over the sphere". These questions were intensively studied during the whole XX century, but several challenging and important problems remain unsolved. The theory has two different generalizations to higher dimensions: quasiregular mappings and holomorphic curves.

The proposer intends to continue his study of geometric questions in the theory of meromorphic functions, using the new techniques developed in his previous work. The main directions of proposed research are following.

Questions related to Bloch's theorem and the Type Problem of a simply connected Riemann surface, especially the relations between the conformal type of a surface and its integral curvature.

Relations between asymptotic properties of a meromorphic function in the plane and geometry of the Riemann surface of the inverse function, and related questions of value distribution of entire functions of finite order.

Problems of geometric function theory arising in real algebraic geometry.

Generalization of results of geometric function theory to quasiregular maps in spaces of arbitrary dimension.

Normality criteria for families of holomorphic curves in projective spaces.

RESULTS FROM PRIOR NSF SUPPORT

The research was titled "Meromorphic Functions and Holomorphic Curves", and was supported by NSF grant DMS–9800084 (Funding period: June 1, 1998 – May 31 2001). The amount of support was $88,589. The results of this research are contained in the 13 papers listed below (10 published, 2 accepted for publication, and one preprint, submitted to a journal).

1. (with M. Bonk) Covering properties of meromorphic functions, negative curvature and spherical geometry, *Annals Math.* 152 (2000), 1-42.
2. (with M. Bonk) Surfaces singulières de fonctions méromorphes, *C.R. Acad. Sci. Paris,* 329, (1999), 953–955.
3. (with M. Bonk) Schlicht regions for entire and meromorphic functions, *J. Analyse Math.* 77, 1999, 69–104.

For a meromorphic function $f : \mathbf{C} \to \bar{\mathbf{C}}$, considered as a map from the complex plane to the Riemann sphere, we define the Bloch radius $B(f)$ as the least upper bound of spherical radii of spherical discs, where continuous branches of the inverse f^{-1} exist. (The length element on the Riemann sphere $\bar{\mathbf{C}}$ is $2|dz|/(1+|z|^2)$).

In the first paper we prove that $B(f) \geq \arccos 1/3 \approx 70°30'$ for every non-constant meromorphic function f. This estimate is best possible, and equality holds for elliptic Weierstrass functions with hexagonal lattices. This is the first time that a hexagonal pattern is shown to be globally extremal for a Bloch-type problem. Earlier Ahlfors [1] proved $B(f) \geq \pi/4$, and Pommerenke [46] improved this to $B(f) \geq \pi/3$.

If all critical points of a meromorphic function f are multiple, then $B(f) \geq \pi/2$, and this is best possible. Earlier D. Minda [41] proved weaker estimates, depending on the multiplicity, and Pommerenke [46] proved $B(f) \geq \pi/2$ for locally univalent meromorphic functions.

The second paper contains an announcement of these results with an outline of proofs, and the third one some preliminary results, and the estimate $B(f) \geq \pi/2$ for non-constant *entire* functions, which is also best possible.

The proofs use a new geometric technique, based on bi-Lipschitz deformation of certain "singular surfaces", associated with meromorphic functions.

4. Bloch radius, normal families and quasiregular mappings, *Proc. Amer.*

Math. Soc., 128 (2000), 557–560.

The classical theorem of Bloch is extended to quasiregular maps in any dimension. Let $\mathbf{U}$ be the unit ball in $\mathbf{R}^n$. For a K-quasiregular map $f : \mathbf{U} \to \mathbf{R}^n$, let $B_e(f)$ be the least upper bound of radii of Euclidean balls, where continuous branches of the inverse f^{-1} exist. It is proved that for any given $M > 0$, the family of all such maps with $B(f) < M$ is a normal family.

When $n = 2$ this is equivalent to Bloch's theorem (without an explicit constant). For arbitrary $n \geq 3$, S. Bochner [8] obtained in 1946 a similar result under the additional assumption that the functions f are harmonic.

Qualitative versions of the results from the papers 1–3 for arbitrary dimension are also obtained.

5. A Picard type theorem for holomorphic curves, *Periodica Mathematica Hungarica*, 38, 1-2 1999, 39–42.

A simple direct proof of the following result is given: Suppose that $2n+1$ hypersurfaces in n-dimensional complex projective space $\mathbf{P}^n$ have the property that every $n+1$ of them have empty intersection. Then every holomorphic map $f : \mathbf{C} \to \mathbf{P}^n$, omitting these hypersurfaces, is constant. This result was essentially known, but the proposer hopes, that the new proof given in the paper will permit to give explicit estimates of the Royden function [33, Ch. IV] in the complement of the hyperplanes. Previous proofs known to the proposer use a form of a rescaling principle, and thus do not lead to effective estimates.

6. (with A. Atzmon and M. Sodin) Spectral inclusion and analytic continuation, *Bull. London Math. Soc.*, 31 (1999), 722–728.

We give a necessary and sufficient condition for the spectrum of an element of a Banach algebra to belong to a given compact set $K \subset \mathbf{C}$. It is formulated in terms of certain "generalized Faber polynomials", associated with K. When K is connected, a similar result was obtained by V. Havin in [29]. Applications to spectral theory are given. We also use these polynomials to obtain a criterion for a germ of an analytic function at ∞ to have an analytic continuation to $\mathbf{C}\backslash K$.

7. (with W. Hayman) On the length of lemniscates, *Michigan Math. J.*, 46, 1999, 409–415.

The following problem was proposed by Erdős, Herzog and Piranian in 1958, and since then it was repeated in several lists of unsolved problems, for example in [30, Probl. 4.10].

What is the maximal length of the level set $E(P) = \{z : |P(z)| = 1\}$ among all monic polynomials P of a given degree d? It is conjectured that $P(z) = z^d + 1$ is extremal; the length of its level set is $2d + o(1)$ when $d \to \infty$. The first upper estimate due to Pommerenke [45] was quadratic with respect to d. Recently P. Borwein [9] proved the upper estimate $68.32d$. Using a different method, we improve this to $9.17d$. But the main result of the paper is the following property of extremal polynomials: all critical points of such polynomial belong to $E(P)$. This property reduces the number of parameters of the problem by a factor of 2. In particular, it follows, that for $d = 2$ the polynomial $P(z) = z^2 + 1$ is indeed extremal.

8. (with W. Hayman) Univalent functions of fast growth with gap power series, *Math. Proc. Camb. Phil. Soc.*, 127 (1999), 525–532.

In 1953 Hayman proved the following theorem about normalized univalent functions in the unit disc $f(z) = z + a_2 z^2 + \ldots$. There always exists a limit

$$\alpha(f) = \lim_{n \to \infty} \frac{|a_n|}{n} \in [0, 1],$$

with $\alpha(f) = 1$ if and only if $|a_n| = n$ for all n. (This result does not follow from the the proof of Bieberbach's conjecture). We tried to test how precise this theorem is, by considering univalent functions with gap power series, that is $a_n = 0$ for infinitely many n. Hayman's theorem gives for such functions $\alpha(f) = 0$, and the question is whether one can say more about the order of growth of coefficients in this case.

We proved the following. Given arbitrary sequence $(\epsilon_n) \to 0$ and arbitrary sequence of intervals (E_k), there exists a normalized univalent function in the unit disc, for which $a_n = 0,\ n \in E_k$ for infinitely many k, and

$$|a_n| \geq \epsilon_n n, \quad n \in E_k,$$

for infinitely many k. So the Taylor series of a univalent function can have arbitrarily large gaps, and at the same time the coefficients may grow on some intervals as fast as Hayman's theorem permits.

9. (with M. Bonk) Uniformly hyperbolic surfaces, Accepted in *Indiana Univ. Math. J.*, 2000.

This paper is related to our work from the papers 1–3. We consider open simply connected surfaces with (possibly singular) Riemannian metrics of non-positive curvature, and prove that several different conditions of "uniform hyperbolicity" are equivalent. Among these conditions are: a linear isoperimetric inequality, hyperbolicity in the sense of Gromov, upper bounds for derivatives of conformal maps from the hyperbolic plane to the surface, and the condition that the integral curvature of every compact disc of a fixed radius on the surface is bounded from above by a negative constant. This gives a geometric generalization of the classical Bloch theorem, as well as some new inequalities for Bloch's functions and for univalent functions.

10. (with W. Hayman) On a conjecture of Danikas and Ruscheweyh, Accepted in *Rendiconti Lincei, Mat. e Appl.*, 2000.

We construct a counter-example to a univalence criterion, recently conjectured by Danikas and Ruscheweyh in [14]

11. (with W. Bergweiler) Entire function of slow growth whose Julia set coincides with the plane, *Ergod. Theory and Dyn. Syst.* 20 (2000), 1-6.

Answering a question of I. N. Baker, we construct examples of transcendental entire functions of arbitrarily slow growth, whose Julia sets coincide with the plane. Arbitrarily slow growth means that $\log|f(z)| \leq O(\phi(|z|)\log|z|)$, as $z \to \infty$, where $\phi > 0$ is a given function, tending to ∞. Methods known before permitted to handle only special classes of entire functions, which do not contain functions of very slow growth.

12. (with A. Gabrielov) Rational functions with real critical points, Preprint MSRI, 2000-002, Submitted to a journal.

It is a classical result of enumerative geometry, that given $2d-2$ lines in general position in d-dimensional complex projective space $\mathbf{P}^n$, there exist

$$u_d := \frac{1}{d}\begin{pmatrix} 2d-2 \\ d-1 \end{pmatrix}, \quad \text{the } d\text{-th Catalan number}, \tag{1}$$

of projective subspaces of codimension 2, intersecting all these lines.

If these given lines are real, the codimension 2 subspaces intersecting all of them are not necessarily real. In real algebraic geometry, one is interested in geometric conditions, which imply that a problem of enumerative geometry with real data has real solutions (see, for example, [25]). Besides an intrinsic interest, this question is important for control theory of linear systems by static output feedback [10, 51]. The following specific conjecture about the problem stated above, was made by B. and M. Shapiro.

The *rational normal curve* of degree d is the map $E : \mathbf{P}^1 \to \mathbf{P}^n$, given in the homogeneous coordinates by $E(z) = (1 : z : \ldots : z^d)$. B. and M. Shapiro conjectured that if $2d - 2$ lines, tangent to E at real points are given, then all u_d codimension 2 subspaces, which intersect these lines, are real.

We proved this conjecture. It turns out to be equivalent to the following statement about rational functions: if all critical points of a rational function f belong to a circle C on the Riemann sphere, then the image $f(C)$ of this circle is also a circle.

The following information was known before: a) Given an integer d and a circle C *there exist* configurations of $2d - 2$ points on C, so that every rational function of degree d with these critical points maps C onto a circle [52]. b) The conjecture was tested numerically for $d \leq 9$ [57].

13. Ahlfors' contribution to the theory of meromorphic functions, in: Lectures in the Memory of Lars Ahlfors, Israel Math. Conf. Proc. 14, AMS, 2000, 41-63.

This is an expanded version of the proposer's lecture in Technion.

Human resources development. The proposer currently advises two PhD students. B. Oh has passed his qualifying examinations in 1999, and now he is working on conformal type criteria for simply connected singular surfaces. S. Merenkov passed his qualifying examinations in summer 2000. Recently he extended a theorem due to the proposer on the characterization of Riemann surfaces by their semigroups of endomorphisms [17] to domains in $\mathbf{C}^n$.

In 1999 the proposer taught a graduate course "Topics in Complex Analysis" and 2 reading courses, partially based on his NSF-sponsored research.

In Spring 1998 the proposer did a joint project with Peter Duselis, then an undergraduate student. We found a rational function f of degree 47 with $B(f) \approx 76°$.

PROJECT DESCRIPTION

Geometric theory of meromorphic functions

1. Questions related to Bloch's theorem and criteria of hyperbolic type.

We recall that the Bloch radius $B(f)$ of a meromorphic function f is defined as the least upper bound of spherical radii of spherical discs, where branches of the inverse f^{-1} exist. The main result in [6] says that

$$B(f) \geq b_0 := \arccos(1/3) \approx 70°30' \tag{2}$$

for every non-constant meromorphic function in the plane. The question of determining the precise estimate of $B(f)$ from below is also interesting for other classes of meromorphic functions. In [5] we considered meromorphic functions f defined on a compact Riemann surface S of given genus g, and obtained lower estimates of $B(f)$ in terms of g. These estimates are asymptotically best possible when g is large. One question which remains unsolved is what happens when $g = 0$, that is for rational functions. Let $B_0 = \inf B(f)$ over the class of all non-constant rational functions f. From (2) follows $B_0 \geq b_0$.

Question 1. *Is it true that $B_0 = b_0$?.*

The known extremal functions for (2) are elliptic $\wp$-functions whose critical points form hexagonal lattices in the plane. Question 1 seems to be related to the existence of finite configurations of points on the sphere, which are close to hexagonal in a certain sense. There are many important extremal problems (encountered in both mathematics and science) about finite configurations of points on the sphere, which lead to similar questions, [40].

We also don't know whether the elliptic $\wp$-functions with hexagonal lattices are the only extremal functions for (2). This leads to

Question 2. *Do there exist meromorphic functions with $B(f) = b_0$, other than elliptic $\wp$-functions with hexagonal lattices?*

It seems plausible that the answer is negative, which would imply an interesting rigidity property of hexagonal lattices.

Question 3. *Is it true that for every non-constant meromorphic function f in the plane, there is an open spherical disc D of spherical radius b_0, such that a branch of the inverse f^{-1} is defined in D?*

Our result [6] only implies that such discs of radius $b_0 - \epsilon$ exist, for every $\epsilon > 0$. Question 3 is related to existence of certain small perturbations of elliptic $\wp$-functions with hexagonal lattices.

The main result of [6] is rather general, because it applies to all non-constant meromorphic functions. On the other hand, it can be considered as a special manifestation of the more general principle, that "negative curvature implies hyperbolicity", which goes back to Ahlfors [2] (see [44] for a survey of recent results). The proposer intends to further explore this principle in the context of geometric function theory.

According to the Uniformization Theorem, for every open simply connected Riemann surface S there exists a conformal homeomorphism $h : D(R) \to S$, where $D(R) := \{z : |z| < R\}$, and $0 < R \leq \infty$. We say that S is of parabolic type if $R = \infty$, and of hyperbolic type if $R < \infty$. If S is given in some explicit way, for example as a surface equipped with a Riemannian metric, a problem arises how the geometric properties of S are related to the properties of the conformal map h, in particular, how to determine the conformal type of S. This is called the Type Problem [3].

In geometric function theory we usually have the following situation. Suppose that V is an abstract topological surface, $\bar{\mathbf{C}}$ is the Riemann sphere, and $g : V \to \bar{\mathbf{C}}$ is an open and discrete map. Then the spherical (Riemannian) metric pulls back from $\bar{\mathbf{C}}$ to V via g, and we obtain a metric space S, which we call a "Riemann surface spread over the sphere", so that the map

$$g : S \to \bar{\mathbf{C}} \tag{3}$$

preserves the lengths of curves. The metric on S uniquely determines the conformal structure, which justifies the term "Riemann surface". If h is a uniformizing map, then $f = g \circ h$ is a meromorphic function. The metric on S is a smooth Riemannian metric of constant Gaussian curvature $+1$ at all points except an isolated set of *singular points* where g is not a local homeomorphism.

Similarly one can consider Riemann surfaces spread over the plane, by pulling back the Euclidean metric from the complex plane.

The main result in [6] can be restated in the following way. *Let S be an open simply connected Riemann surface spread over the sphere, such that the least upper bound of radii of the smooth discs in S is less than b_0. Then S is of hyperbolic type.*

To explain why this is so, one has to consider the integral curvature on S. The Gaussian curvature in the usual sense does not exist at the singular points, so one considers the generalized notion of integral curvature, due to Aleksandrov, Huber and Reshtniak [47, 31]. In what follows we mean by a Riemannian metric on a Riemann surface an intrinsic metric, whose length element is given in local coordinates by $ds = \exp\{-u(z)\}|dz|$, where u is a difference of two subharmonic functions, such that $\exp\{-u(z)\}$ is integrable on rectifiable curves. The integral curvature is the Laplacian Δu.

For the surfaces spread over the sphere, described above, at each singular point, where g has local degree m, the integral curvature has an atom of mass $2\pi(1-m)$. On the smooth part of S the integral curvature is equal to the area. Thus the singular points contribute to the negative part of the integral curvature, while the smooth part of the surface contributes to the positive part.

The condition that there are no smooth discs of radius more than $b_0 - \epsilon$ implies that the singular points are relatively dense on the surface, so for large pieces of the surface, the negative contribution from singular points dominates the positive contribution from the smooth part, so the integral curvature of such pieces is negative. We show in [6] that this implies hyperbolic type. In fact this implies a stronger property, known as uniform hyperbolicity or Gromov hyperbolicity [27].

Ahlfors' generalization of the Schwarz lemma [2] implies the following: *every simply connected surface, whose Gaussian curvature is bounded from above by a negative constant, is of hyperbolic type.* Our result in [6] shows that under certain circumstances one can replace the condition that curvature is bounded from above by a negative constant at every point, by a weaker condition, involving only integral curvature of large pieces of the surface.

One natural conjecture is the following

Conjecture 1. *Let S be an open simply connected surface with a Riemannian metric, whose Gaussian curvature is bounded from above. Suppose that there exist $\epsilon > 0$ and $R > 0$, such that whenever a disc $D(a, R) \subset S$ has compact closure, its integral curvature is at most $-\epsilon$. Then the surface S is* *of hyperbolic type.*

If this is true, one can probably strengthen the conclusion to "uniform hyperbolicity" as in [6, 7].

In [7] the uniform hyperbolicity was proved under the assumptions of Conjecture 1 and the additional assumption that the curvature is non-positive. When applied to Riemann surfaces spread over the plane, this result is equivalent to Bloch's theorem.

Another conformal type criterion of this sort was conjectured by R. Nevanlinna [43, Ch. XII, §1]. He considered a special class of Riemann surfaces spread over the sphere, which are ramified only over a finite set. We will call such surfaces *critically finite*. Nevanlinna defines a quantity V, which he calls the average ramification of such a surface, and conjectures that $V = 2$ implies parabolic type and $V > 2$ hyperbolic. The first part of this conjecture was disproved in [56], but the second part, when restated in terms of integral curvature leads to the following

Conjecture 2. *Duppose that S is complete. If for some $a \in S$, $\epsilon > 0$ and $R_0 > 0$ we have*

$$\text{integral curvature}\,(D(a,R)) \leq -\epsilon\,\text{area}\,D(a,R), \quad \textit{for} \quad R > R_0,$$

then S is of hyperbolic type.

In this form the conjecture applies to arbitrary complete surfaces with a Riemannian metric. If it is true, one can probably extend it to simply connected surfaces, which are not necessarily complete.

2. Critically finite Riemann surfaces spread over the sphere and the Littlewood phenomenon

Critically finite surfaces were defined in the previous subsection. It is worth mentioning that meromorphic functions in the plane, corresponding to this class of surfaces play an important role in the general theory of meromorphic functions [43, 55] as well as in holomorphic dynamics [4, 23, 22]. Teichmüller proved in [54] that the conformal type of a critically finite surface spread over the sphere is determined by its topology. It is natural to ask, to what extent the other properties of the meromorphic function $f = g \circ h$, where h is the uniformizing map of S, are defined by topological properties of the map g in (3).

Suppose that S is known to be of parabolic type, so that f is a meromorphic function in the plane. Can the order (of growth) of f be determined from topological properties of g? An example by Künzi [32] gives a negative answer to this question for critically finite meromorphic functions. However it seems plausible that the answer may be positive for the case of entire functions.

Question 4. *(A. Epstein). Suppose that f_1 and f_2 are critically finite entire functions, and $f_1 = \phi \circ f_2 \circ \psi$, where ϕ and ψ are homeomorphisms. Does this imply that f_1 and f_2 have the same order?*

Examples are known, which show that the answer may be negative for entire functions that are not critically finite [59]. It turns out that the following conjecture implies a positive answer to Question 4.

Conjecture 3. *For every critically finite entire function f of finite order there exists a set $E \subset \mathbf{C}$, which has zero density, that is*

$$\operatorname{area} E \cap \{z : |z| \leq r\} = o(r^2),\ r \to \infty, \tag{4}$$

and such that for every $a \in \mathbf{C}$, almost all solutions of the equation $f(z) = a$ belong to E.

"Almost all" means here "all but finitely many". Conjecture 3 is not so implausible as it may seem at the first glance. In fact something very similar is known to be true even for arbitrary entire functions of finite order (but not for meromorphic functions!). This is called Littlewood's phenomenon. Namely, for every entire function f of finite order $\lambda > 0$ there exists a set E with the property (4), and such that for every $a \in \mathbf{C}$ most of the solutions of the equation $f(z) = a$ belong to E. Here "most" means that

$$n(r, a) = n(r, a, E) + O(r^{\lambda - \epsilon}),\ r \to \infty, \quad \text{where} \quad \epsilon > 0. \tag{5}$$

This result was derived by Littlewood [36] from one conjectural inequality for polynomials. It was proved by M. Sodin and the proposer in [24] with a worse error term $o(r^\lambda)$, and after that Lewis and Wu [35] proved Littlewood's conjecture for polynomials, thus establishing (5). Later Lewis [34] found a generalization of (5) to entire functions of infinite order, and even to quasiregular maps in $\mathbf{R}^n$.

It is not known whether the error term in (5) can be improved to $O(1)$, even for arbitrary entire functions, though this is very unlikely in view of examples in [16]. On the other hand, the assumption that an entire function is critically finite might imply (5) with the error term $O(1)$, which is Conjecture 3.

3. Meromorphic functions with real critical points

In [21] we parametrized an interesting class of rational functions, namely those whose all critical points belong to the real line. This paper generates several challenging questions which the proposer plans to investigate.

Two meromorphic functions f and g will be called equivalent, $f \sim g$, if $f = l \circ g$ with a fractional-linear transformation l. By the chain rule, equivalent functions have the same critical points. The main result in [21], conjectured by B. and M. Shapiro, is the following

Theorem 1. *Let f be a rational function whose all critical points are real. Then f is equivalent to a real rational function.*

This is almost trivial if f is a polynomial. The reason is that any two polynomials with the same critical points are equivalent. This is not so for rational functions: given a generic configuration of $2d-2$ points on the Riemann sphere there exist u_d classes of rational functions of degree d with these critical points, where u_d are the Catalan numbers defined in (1). This result was proved in [26], using a reduction of the problem to enumerative geometry in projective space.

Writing a rational function as $f = f_1/f_0$ we arrive to the following reformulation: *Suppose that the Wronskian determinant $W(f_0, f_1)$ of two polynomials has only real zeros. Then there exists a linear transformation with constant coefficients, which transforms the vector (f_0, f_1) into a vector of real polynomials.*

Conjecture 4. (B. and M. Shapiro, [52, 53]) *The same is true for arbitrary number of polynomials.*

Our method of proof of Theorem 1 does not extend to a proof of this conjecture. The proposer thinks that an approach based on linear differential equations with regular singular points may be more useful. As a first step one has to find a new proof of Theorem 1, based on this approach.

The set of equivalence classes of rational functions of degree d is parametrized by the Grassmanian $G(2,d)$, and the set of unordered $2d-2$-tuples of points in $\bar{\mathbf{C}}$ is parametrized by $\mathbf{P}^{2d-2}$. Thus we have a map $W : G(2,d) \to \mathbf{P}^{2d-2}$ sending a class of rational functions into its critical set. Our Theorem 1 implies that this map W is unramified over the set of *real* $2d-2$-tuples. It is an important problem (stated in [26]) to find exactly the ramification locus of W. The proposer plans to study this question.

Another way to generalize Theorem 1 is the following.

Conjecture 5. *Suppose that $f : S_1 \to S_2$ is a holomorphic map between two compact Riemann surfaces, and S_1 has an anti-conformal involution τ_1 which leaves all critical points of f fixed. Then there is an anti-conformal involution τ_2 of S_2, such that $f \circ \tau_1 = \tau_2 \circ f$.*

Theorem 1 corresponds to the case when $S_1 = S_2 = \bar{\mathbf{C}}$. The next case to consider is $S_2 = \bar{\mathbf{C}}$ and S_1 is of genus zero. The proposer hopes that the method of proving Theorem 1 can be extended to this case.

The main new ingredient of the proof of Theorem 1 is a parametrization of the class of real rational functions whose all critical points are real. The similar class of polynomials permits a natural closure, which is a class M of entire functions, studied by MacLane [37] and Vinberg [58]. This class of entire functions is useful in many questions of analysis, see for example [38].

Thus it is interesting to find out, what is the transcendental counterpart of the class of real rational functions whose critical points are real. Here are two questions to begin with:

Question 5. *Suppose that all critical points of a meromorphic function f of genus zero belong to the real line. Does it follow that f is equivalent to a real meromorphic function?*

Question 6. *Which real meromorphic functions whose all critical points are real can be obtained as limits of similar rational functions?*

4. Explicit constants in multi-dimensional extensions of Geometric Function Theory.

A continuous map f from a region in R^n to R^n is called K-quasiregular if it belongs to the Sobolev class W_n^1, and the following inequality holds a. e.

$$\|f'\|^n \leq K J_f,$$

where $\| \, \|$ is the usual L^2 norm, and J_f is the Jacobian determinant. A quasiregular map between Riemannian manifolds of the same dimension is defined using local coordinates.

The theory of K-quasiregular maps is regarded a reasonable generalization of geometric function theory to higher dimensions [48, 49, 39]. For example, quasiregular maps are open and discrete, and Picard's theorem has a (very non-trivial) generalization for them [50].

In [19] several versions of the Bloch theorem were proved for quasiregular mappings, both for spherical and Euclidean metrics. The simplest of them is for the maps of a sphere into itself.

Let S^n be the unit n-dimensional sphere, equipped with the standard spherical metric, and $f : S^n \to S^n$ a non-constant K-quasiregular map. Then there exists a continuous branch of the inverse f^{-1} defined in some disc $D(a, R) \subset S^n$, of radius $R \geq R_0(K, n)$, where R_0 depends only on K and n.

When $n = 2$, R_0 can be chosen independent of K, in fact we proved in [5] that one can take $R_0 = b_0$, the constant from (2). On the other hand, for $n \geq 3$, examples show that R_0 does indeed depend on K. A major drawback of the existing proof of this theorem is that it is a pure existence theorem: it gives no explicit value for R_0. The proposer intends to further investigate this question, and try to find an explicit constant. The problem is related to the question, how large the ramification locus of a quasiregular map in dimension ≥ 3 can be. Unlike in dimension 2, very little is known about this.

Another high-dimensional counterpart of the Geometric Function Theory is the theory of holomorphic curves in projective space. The following result due to Dufresnoy [15] can be considered as an extension of Montel's normality criterion.

Let f be a holomorphic map from the unit disc to the projective space of dimension n, omitting $2n+1$ hyperplanes in general position. Then $\|f'\| < C$, where $C > 0$ depends only on the omitted hyperplanes. The norm of the derivative is measured using the Poincaré metric in the unit disc, and the Fubini–Study metric in the projective space.

All existing proofs of this result for $n \geq 3$ use some form of a rescaling principle (a. k. a. Brody's lemma [33, Ch. III]), which does not give any information about the constant C. Only in dimension 2 effective estimates are available [13, 28]. For higher dimensions, effective estimates were obtained in [12] under much stronger assumption that f omits $2^n + 1$ hyperplanes. The proposer intends to obtain an explicit estimate, using his potential-theoretic methods from [20]. An alternative approach can be based on obtaining explicit estimates in Cartan's theorem, mentioned in the next section.

5. Cartan's Conjecture

The generalization of Picard's theorem to holomorphic curves we discussed so far says that a holomorphic curve $\mathbf{C} \to \mathbf{P}^n$, omitting $2n+1$ hyperplanes in general position, is constant. There is a more refined version, saying that a holomorphic curve, omitting $n+2$ hyperplanes in general position, is linearly degenerate. To make the statement about degeneracy more precise, we introduce convenient homogeneous coordinates. Then the result, which is due to Emil Borel, is the following:

Let $f_1, \ldots, f_p$ be zero-free entire functions, which satisfy

$$f_1 + \ldots + f_p = 0 \tag{6}$$

Then the set of indices $\{1, \ldots, p\}$ can be partitioned into subsets I, such that the functions, whose indices are in the same subset, are proportional and $\sum_{j \in I} f_j = 0$ for every I.

When $p = 3$, this is equivalent to Picard's theorem.

The problem is to find a normality criterion related to Borel's theorem in the same way as Montel's normality criterion is related to Picard's theorem. Unlike for the similar problem considered in subsection 4, this cannot be done with a rescaling argument, and the problem is much more subtle. This problem was studied for the first time by Bloch in 1925. His work was continued by H. Cartan in 1928 [11]. This work of Cartan is one of the central results in complex hyperbolic geometry (see, for example [33]).

Let us denote by $U(R),\ R > 0$ the set of all holomorphic functions without zeros in $D(R) = \{z : |z| < R\}$. We make the following assumptions:
(A). *Let Σ be an infinite sequence of p-tuples $f = (f_1, \ldots, f_p)$ such that $f_j \in U(R)$ for every $j = 1, \ldots, p$ and every $f \in \Sigma$, and the condition* (6) *is satisfied for every $f \in \Sigma$.*

The following conjecture was made by Cartan [11]: Under the assumptions (**A**) one can find an infinite subsequence $\Sigma' \subset \Sigma$ with the following

property: the set of indices $\{1,\ldots,p\}$ can be partitioned into subsets I such that for $f \in \Sigma'$ and every I we have
(i) f_j/f_k are uniformly bounded on compact subsets of $D(R)$ for some $k \in I$ and all $j \in I$;
(ii) $\sum_{j\in I} f_j/f_k \to 0$ uniformly on compact subsets of $D(R)$, as $f \in \Sigma'$ and $f \to F$. Here F is the filter consisting of complements of finite subsets of Σ', and $k \in I$ is the index for which (i) holds. A subset $I \subset \{1,\ldots,p\}$, satisfying (i) and (ii), will be called a C-class.

If $p = 3$, Cartan's conjecture is equivalent to Montel's normality criterion. Cartan [11] proved his conjecture for $p = 4$ and obtained partial results for arbitrary p. He proved that under the assumptions (**A**) either the whole set of indices $\{1,\ldots,p\}$ makes a C-class, or there exist at least two disjoint C-classes.

In [18] the proposer constructed a counterexample to Cartan's conjecture for every $p \geq 5$. In fact the conjecture remains wrong even if one replaces (ii) by a weaker condition that every I contains at least two elements. In the same paper the proposer found, how to modify Cartan's conjecture so that it becomes true for $p = 5$.

Conjecture 6. *under the assumptions* (**A**) *there exists an infinite subsequence* $\Sigma' \subset \Sigma$ *with the following property: the set of indices* $\{1,\ldots,p\}$ *can be partitioned into subsets I such that for* $f \in \Sigma'$ *and every I we have* (i) *and* (ii) *in the disc* $D(kR)$*, where* $k = k_p < 1$ *is a constant depending only on* p*.*

Conjecture 6 has been proved in [18] for $p = 5$ with $k_5 = 1/64$. The proposer believes that his potential-theoretical arguments combined with Nevanlinna theory as in [18] will eventually lead to a complete proof of Conjecture 6. He already has a partial result for arbitrary p, but with an additional restriction of the sequence Σ.

Cartan's proof of his results is complicated, and all attempts to simplify it failed so far. It seems possible to extract effective estimates from this proof, and this was done by P. Hall for $p = 4$ in [28]. However Hall's estimates are weaker than expected, so the question deserves further investigation.

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