Dynamics of a three-dimensional analog of the sine function

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This talk is based on a joint work with Walter Bergweiler. We construct a dynamical system in $\mathbf{R}^n$ which displays a strong form of the "Karpińska paradox" [2, 3].

A set $H \subset \mathbf{R}^n$ is called a *hair* if there exists a continuous injective map $\gamma : [0, \infty) \to \mathbf{R}^n$ such that $\gamma(t) \to \infty$ as $t \to \infty$, and $\gamma([0, \infty)) = H$. The point $E = \gamma(0)$ is called the *endpoint* of the hair.

Theorem. *The space $\mathbf{R}^n, n \geq 2$ can be represented as a union of hairs with the following properties. The intersection of any two hairs is either empty or consists of their common endpoint, and the union of hairs without endpoints has Hausdorff dimension one.*

Analogous decomposition of $\mathbf{R}^2$ was obtained by Schleicher [3] who used the dynamics of an entire function of the sine family.

To obtain the result in $\mathbf{R}^n$ we use a quasiregular map which generalizes the sine map of the complex plane in the similar way to Zorich's generalization of the exponential map [4].

To simplify notation, we describe our construction in $\mathbf{R}^3$.

We denote by $H_{\geq 0}$ and $H_{\leq 0}$ the upper and lower half-spaces in $\mathbf{R}^3$.

Let $h : Q \to U, \quad h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2), h_3(x_1, x_2))$ be a bi-Lipschitz map from the square

$$Q = [-1, 1] = \{(x_1, x_2) \in \mathbf{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$$

onto the upper hemisphere

$$U = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq 0\}.$$

We first define our map F in the semi-cylinder

$$Q \times [1, \infty) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : |x_1| \leq 1, |x_2| \leq 1, x_3 \geq 1\}$$

by the formula

$$F(x_1, x_2, x_3) = (e^{x_3} h_1(x_1, x_2), e^{x_3} h_2(x_1, x_2), e^{x_3} h_3(x_1, x_2)) .$$

Then we extend F to the cube $Q \times [0, 1]$, so that extended function is bi-Lipschitz and maps this cube onto the upper half of the ball

$$\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq e^2, \ x_3 \geq 0\},$$

and the resulting map $F : Q \times [0, \infty) \to H_{\geq 0}$ is bi-Lipschitz on compact subsets. Then our map F is extended to a map $\mathbf{R}^3 \to \mathbf{R}^3$ by symmetry, using reflections in the faces of the semi-cylinders and in the plane $x_3 = 0$.

The resulting map is quasiregular, bi-Lipschitz on compact subsets, and the infinitesimal distortion tends to infinity as $x \to \infty$. Finally we multiply F by a positive constant so that the map $f = \lambda F$ is uniformly expanding.

The dynamics of this map f is studied using a symbolic dynamics similar to that introduced by Devaney and Krych [1] for the exponential map of the complex plane.

We put $S = \mathbf{Z} \times \mathbf{Z} \times \{-1, 1\}$ and for $r = (r_1, r_2, r_3) \in S$ define the set

$$T(r) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : |x_1 - 2r_1| \leq 1, |x_2 - 2r_2| \leq 1, r_3 x_3 \geq 0\}.$$

If $r_1 + r_2 + (r_3 - 1)/2$ is even then f maps $T(r)$ bijectively onto $H_{\geq 0}$. If $r_1 + r_2 + (r_3 - 1)/2$ is odd then f maps $T(r)$ bijectively onto $h_{\leq 0}$.

For a sequence $\underline{s} = (s_k)_{k \geq 0}$ of elements of S we put

$$H(\underline{s}) = \{x \in \mathbf{R}^3 : f^k(x) \in T(s_k) \quad \text{for all} \quad k \geq 0\}.$$

Evidently $\mathbf{R}^3 = \sum_{\underline{s} \in \underline{S}} H(\underline{s})$, where $\underline{S}$ is the set of all sequences $\underline{s}$ with elements in S for which $H(\underline{s})$ is not empty.

Proposition 1. *If $\underline{s} \in \underline{S}$ then $H(\underline{s})$ is a hair.*

For $\underline{s} \in \underline{S}$ we denote by $E(\underline{s})$ the endpoint of $H(\underline{s})$.

Proposition 2. *If $\underline{s}' \neq \underline{s}''$ then $H(\underline{s}') \cap H(\underline{s}'') = \emptyset$ or $H(\underline{s}') \cap H(\underline{s}'') = E(\underline{s})$.*

Proposition 3. *The Hausdorff dimension of the set*

$$\bigcup_{\underline{s}\in\underline{S}} H(\underline{s})\backslash\{E(\underline{s})\}$$

equals to one.

Theorem 1 follows from these three propositions.

References

[1] R. Devaney and M. Krych, Dynamics of $\exp(z)$, Ergodic theory and dynamical systems, 4 (1984) 35–52.

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[3] D. Schleicher, The dynamical fine structure of iterated cosine maps and a dimension paradox, Duke Math. J., 136 (2007) 343-356.

[4] V. Zorich, A theorem of M. A. Lavrent'ev on quasiconformal space maps, Math USSR Sbornik, 3 (1967) 389–403.