Linear ordinary differential equations

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A linear ordinary differential equation of order n is

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = f(x), \tag{1}$$

where $a_0, \ldots, a_{n-1}$ and f are given functions defined on some interval, y is the unknown function, and $y^{(k)} = d^k y/dx^k$. Functions a_k are called the *coefficients*.

If $f = 0$ the equation is called *homogeneous*.

Solutions of a homogeneous equation form a vector space: if $y_1, \ldots, y_k$ are solutions then any linear combination of them, $c_1y_2 + c_2y_2 + \ldots + c_ky_k$ is also a solution.

The difference of any two solutions of a non-homogeneous equation is a solution of the homogeneous one (with the same LHS). So to find the general solution of (1) it is sufficient to find *one* solution of (1) and the general solution of the homogeneous equation. Then the general solution of (1) will be the sum of a particular solution and the general solution of the homogeneous equation.

Cauchy problem for equation (1) is to find a solution which satisfies the initial conditions:

$$y(x_0) = a_0, \quad y'(x_0) = a_1, \quad \ldots, y^{(n-1)}(x_0) = a_{n-1}.$$

The number of conditions is the same as the *order* n of the equation.

Existence and uniqueness theorem. *If the coefficients and f are continuous functions on some interval I, and x_0 is a point in I, then Cauchy problem has a unique solution on I for any initial conditions $y_0, \ldots, y_{n-1}$.*

The interval I can be finite or infinite.

It follows from this theorem that the vector space of solutions of a homogeneous equation on a given interval has dimension n (the order of the equation). So to find the general solution, it is sufficient to find n linearly independent ones. If $y_1, y_2, \ldots, y_n$ are linearly independent solutions, then the general solution is

$$y = c_1 y_2 + \ldots + c_n y_n.$$

Very few equations of the form (1) can be solved explicitly, that is in terms of elementary functions. The most important such type is equations with *constant coefficients*. Consider first a homogeneous equation with constant coefficients:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = 0. \tag{2}$$

Let us look for a solutions of the form $y(x) = e^{\lambda x}$. Substituting this to the equation and dividing on $e^{\lambda x}$ we obtain a polynomial equation for λ:

$$\lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0 = 0. \tag{3}$$

By the fundamental theorem of algebra, this polynomial equation always has solutions. Let us show that distinct roots of this equation give linearly independent solutions. Let us take distinct roots $\lambda_1, \ldots, \lambda_m$. We have to show that

$$c_1 e^{\lambda_1} + \ldots + c_m e^{\lambda_m} \equiv 0 \tag{4}$$

implies that $c_1 = \ldots = c_m = 0$. To do this we differentiate (4) $m - 1$ times and consider the resulting system of linear equations with respect to c_k (together with the original equation (4)):

$$\sum_{k=1}^{m} c_k \lambda_k^j e^{\lambda_k x} = 0, \quad j = 0, \ldots, m-1.$$

Plug $x = 0$, then the determinant of the system will be

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \ldots & \lambda_m^{m-1} \end{vmatrix}.$$

This is called the *Vandermonde determinant*, and we know from Linear Algebra that it is not zero when all λ_j are distinct.

A generic polynomial (3) will have n distinct (complex) roots, so our method gives n linearly independent solutions of (2), and thus the general solution.

If this polynomial has a multiple root λ, then one can show that together with $e^{\lambda z}$, there is another solution $xe^{\lambda x}$, which is linearly independent from the rest. In general, if λ is a root of multiplicity m of (3), then one can find m linearly independent solutions,

$$e^{\lambda x}, xe^{\lambda x}, \ldots, x^{m-1}e^{\lambda x},$$

which correspond to this root λ. Taking them all together, we obtain a basis of solutions of (2).

Example 1. $y'' + a^2 y = 0$, where a is real. The characteristic equation is $\lambda^2 + a^2 = 0$. It has roots $\pm ia$. We obtain two linear independent solutions e^{iax}, e^{-iax}. To obtain two real linearly independent solutions we take linear combinations

$$y_1(x) = (e^{iax} + e^{-iax})/2 = \cos ax, \quad \text{and} \quad y_2(x) = (e^{iax} - e^{-iax})/(2i) = \sin ax.$$

So the general real solution is $y(x) = c_1 \cos ax + c_2 \sin ax$.

Non-homogeneous equation. There are three main methods of finding one solution. a) guessing, b) the method of variation of constants, and c) Laplace transform.

I will explain the method of variation of constants on a simple example

$$y'' + a^2 y = f(x). \tag{5}$$

The egeneral method is to look for a solution y *with has the same form as the general solution of the homogeneous equation, but with* **non-constant coefficients***, and try to find these coefficients.* So for a second order equation, we look for y in the form

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x),$$

where y_1, y_2 are linearly independent solutions of the homogeneous equation. We have

$$y' = c_1 y_1' + c_2 y_2' + (c_1' y_1 + c_1' y_2). \tag{6}$$

We *impose* the additional condition

$$c_1' y_1 + c_2' y_2 = 0, \tag{7}$$

so that the expression in parentheses in (6) vanishes. Then

$$y'' = c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2'.$$

Substituting this to (5) and using that y_1, y_2 satisfy $y'' + a^2 y = 0$, we see that many things cancel, and what remains is

$$c_1' y_2' + c_2' y_2' = f. \tag{8}$$

So we have two equations (7), (8) with two unknown functions c_1' and c_2'. Solving them by Cramer's Rule, we obtain

$$c_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2\prime \end{vmatrix}}, \quad c_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2\prime \end{vmatrix}}.$$

Using $y_1(x) = \cos ax$ and $y_2(x) = \sin ax$, we compute the determinants and obtain

$$c_1'(x) = -\frac{1}{a} f(x) \sin ax, \quad c_2' = \frac{1}{a} f(x) \cos ax,$$

and c_1, c_2 can be found by integration. Putting all together we obtain a partial solution of the non-homogeneous equation in the form

$$\begin{aligned} y(x) &= -\frac{1}{a} \cos(ax) \int_{x_0}^{x} f(t) \sin(at) dt + \frac{1}{a} \sin(ax) \int_{x_0}^{x} f(t) \cos(at) dt \\ &= \frac{1}{a} \int_{x_0}^{x} f(t) \sin\left(a(x-t)\right) dt. \end{aligned}$$

You can check that this formula is correct by substituting it to equation (5).