Orthogonal polynomials

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Recall that the space of all polynomials has a basis (in the usual sense), namely $1, x, x^2, \ldots$. However, as it was explained before, in infinite dimensional spaces it is preferable to have a complete orthogonal system, rather then a basis.

We can endow the space of polynomials with various dot products, and find orthogonal bases by the process of *orthogonalization* described in the handout "Sturm-Liouville". In this way we obtain various systems of *orthogonal polynomials*, depending on the dot product.

All our spaces will be of the form $L^2_w(a,b)$ where a, b can be finite or infinite, and w is a positive function on (a,b) which is called the *weight*. So the dot product will be always defined by

$$(f,g)_w = \int_a^b f(x)\overline{g(x)}w(x)dx.$$

Three cases are the most important ones, they are called the *classical orthogonal polynomials*, and we will study these three cases. It will turn out that orthogonal polynomials are eigenfunctions of sertain singular Sturm–Liouville problems.

We will not actually perform orthogonalization in each case, because in all these three cases there exists a simple explicit formula for our orthogonal polynomials. It is called the Rodriguez formula. These three cases bear the name of French mathematicians of 19th century, Legendre, Laguerre and Hermite.

1. Legendre polynomials P_n: $(a,b) = (-1,1),\ w(x) = 1$.

2. Laguerre polynomials L_n^α, where $\alpha > -1$ is an additional parameter: $(a,b) = (0,\infty),\ w(x) = x^\alpha e^{-x}$.

3. Hermite polynomials H_n: $(a,b)=(-\infty,\infty),\ w(x)=e^{-x^2}$.

It is important to notice that polynomials of all degrees indeed belong to the space, that is

$$\|f\|_w^2=\int_a^b |f(x)|^2 w(x)dx<\infty$$

for every polynomial f. Check this in each case.

All these polynomials have some common properties:

There is one orthogonal polynomial for each degree,

The orthogonal polynomial of degree n is orthogonal to *all* polynomials of degree at most $n-1$.

In the Legendre and Hermite cases, orthogonal polynomials of odd degree are odd, and polynomials of even degree are even. This is because in these two cases, the weight w is even.

They all have simple generating functions, and most importantly, satisfy a simple differential equation.

Moreover, each sequence of orthogonal polynomials satisfies a 3-term recurrent relation (with respect to the degree).

Now we state their main properties exactly.

Legendre polynomials P_n.

Definition (Rodriguez formula):

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n=\frac{(2n)!}{2^n(n!)^2}x^n+\ldots.$$

First few:

$$P_0(x)=1,\quad P_1(x)=x,\quad P_2(x)=\frac{1}{2}(3x^2-1),\quad P_2(x)=\frac{1}{2}(5x^2-3x),\ldots.$$

Squared norms:

$$\|P_n\|^2=\frac{2}{2n+1}.$$

Differential equation in Sturm–Liouville form:

$$\big((1-x^2)P_n'(x)\big)'+n(n+1)P_n(x)=0.$$

or, if we open parentheses

$$(1-x^2)P''-2xP'+n(n+1)P=0.$$

Generating function:

$$\sum_0^\infty P_n(x)z^n=\frac{1}{\sqrt{1-xz+z^2}}.$$

Recursion formula:

$$(n+1)P_{n+1}(x)-(2n+1)xP_n+nP_{n-1}(x)=0.$$

Values at the ends:

$$P_n(1)=1,\quad P_n(-1)=(-1)^n.$$

Laguerre polynomials L_n^α.

Definition (Rodriguez formula):

$$L_n^\alpha=\frac{x^{-\alpha}e^{-x}}{n!}\frac{d^n}{dx^n}\left(x^{\alpha+n}e^{-x}\right)=\frac{(-x)^n}{n!}+\dots.$$

First few:

$$L_0^0(x)=1,\quad L_1^0(x)=-x+1,\quad L_2^0(x)=\frac{1}{2}(x^2-4x+2),\dots.$$

Squared norms:

$$\|L_n^\alpha\|^2=\frac{\Gamma(n+\alpha+1)}{n!}.$$

Differential equation in Sturm–Liouville form:

$$\left(x^{\alpha+1}e^{-x}(L_n^\alpha)'\right)'+nx^\alpha e^{-x}L_n^\alpha=0.$$

or, if we open parentheses,

$$x(L_n^\alpha)''+(\alpha+1-x)(L_n^\alpha)'+nL_n^\alpha=0.$$

Generating function:

$$\sum_0^\infty L_n^\alpha(x) z^n = \frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+1}}.$$

Recursion formula:

$$(n+1)L_{n+1}^\alpha(x) + (x-\alpha-2n-1)L_n^\alpha(x) + (n+\alpha)L_n^\alpha(x) = 0.$$

Hermite polynomials H_n.

Definition (Rodrigues formula):

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = 2^n x^n + \dots.$$

First few:

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \dots.$$

Differential equation in Sturm–Liouville form:

$$\left(e^{-x^2} H_n'\right)' + 2ne^{-x^2} H_n = 0$$

or, if we open parentheses,

$$H_n'' - 2xH_n' + 2nH_n = 0.$$

Generating function:

$$\sum_0^\infty H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}.$$

Recursion formula:

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0.$$

Boundary conditions.

Here we state without proofs the appropriate boundary conditions which ensure that our orthogonal polynomials are eigenfunctions.

1. Consider the *Legendre equation*

$$\Big((1-x^2)y'\Big)' + \lambda y = 0. \tag{1}$$

This equation has two singularities: ± 1. If one tries to find a power series solution of the form

$$y(x) = (x-1)^\rho \sum_0^\infty a_k (x-1)^k, \quad a_0 \neq 0,$$

one finds that the characteristic equation has only one root $\rho = 0$, and this indicates that there is one power series solution, while the other linearly independent solution is unbounded near $x = 1$ (in fact its expansion contains logatithm). Similar situation prevails at the other singular point -1.

The condition that Legendre's equation has a non-zero solution which is bounded on *both ends* implies that $\lambda = n(n+1)$ for some integer $n \geq 0$. So one can say that the Sturm–Liouville problem (1) with boundary conditions:

$$y(\pm 1) \quad \text{are both finite}$$

has eigenvalues $\lambda_n = n(n+1)$, $n = 0, 1, 2\ldots$ and eigenfunctions P_n, the Legendre polynomials.

2. For the *Laguerre equation*

$$(x^{\alpha+1}e^{-x}y')' + \lambda x^\alpha e^{-x} y = 0, \tag{2}$$

the power series method applied to the endpoint $x = 0$ shows that there is one solution which is analytic at 0 (=expands into a power series with non-negatiove integer powers), while the other linearly independent solution behaves like $x^{-\alpha}$ when $\alpha \neq 0$ or like $\log x$ when $\alpha = 0$, as $x \to 0$. So when $\alpha \geq 0$, the boundary condition is that the solution stays bounded as $x \to 0$. When $\alpha \in (-1, 0)$ this is relaced by the condition that derivative of the solution stays bounded.

The other boundary condition is that $y \in L^2_w(0, \infty)$. This is the condition on the growth of the solution at infinity.

With these two boundary conditions, problem (2) has eigenvalues $\lambda_n = n$, $n = 0, 1, 2, 3\ldots$, and eigenfunctions are Laguerre polynomials.

3. For the Hermite equation

$$\left(e^{-x^2}y'\right)' + \lambda e^{-x^2}y = 0, \tag{3}$$

the appropriate boundary condition is the requirement that $y \in L^2_w(-\infty, +\infty)$ (this is essentially a growth restriction on both ends), and the problem (3) with this boundary condition has eigenvalues $\lambda_n = 2n$ and eigenfunctions H_n, the Hermite polynomials.

Associated Legendre equation and corresponding BVP

We will also need the differential equation

$$\left((1-x^2)S'\right)' - \frac{m^2 S}{1-x^2} + \lambda S = 0, \tag{4}$$

which is called the *associated Legendre equation.* Here m is a non-negative integer; when $m = 0$ this coinsides with the Legendre equation.

Solutions are obtained by differentiating the Legendre equation (1) m times and setting

$$S(x) = (1-x^2)^{m/2} y^{(m)},$$

where y satisfies Legendre's equation. The details of this somewhat tedious calculation are given on p. 175-176 of the book. So when $y = P_n$ is a Legendre polynomial, we obtain the so-called *associated Legendre functions* (they are not polynomials when m is odd!)

$$P_n^m(x) = (1-x^2)^{m/2} P_n^{(m)}(x) = \frac{(1-x^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n,$$

where we used the Rodrigues formula for Legendre polynomials. Notice that $P_n^m = 0$ for $m > n$. So to each n correspond one Legendre polynomial P_n and m associated Legendre functions P_n^m, $m = 1, \ldots, m$. These associated Legendre functions solve the Sturm-Liouville problem (4) with the bounbdary conditions

$$S(1) = S(-1) = 0.$$

Furthermore,

For each $m \geq 1$, associated Legendre functions $(P_n^m)_{n=m}^{\infty}$ make an orthogonal basis in $L^2(-1,1)$, and

$$\|P_n^m\|^2 = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}.$$