On the “pits effect” of Littlewood and Offord

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Abstract

Asymptotic behavior of the entire functions

\[ f(z) = \sum_{n=0}^{\infty} e^{2\pi i \alpha_n} z^n / n! \]

with real \( \alpha_n \) is studied. It turns out that the Phragmén–Lindelöf indicator of such function is always non-negative, unless \( f(z) = e^{az} \). For special choice \( \alpha_n = \alpha_n^2 \) with irrational \( \alpha \), the indicator is constant and \( f \) has completely regular growth in the sense of Levin–Pfluger. Similar functions of arbitrary order are also considered.

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In [21] Nassif studied (on Littlewood’s suggestion) the asymptotic behavior and the distribution of zeros of the entire function

\[ \sum_{n=0}^{\infty} e^{2\pi i \alpha_n^2} z^n / n! \]

with \( \alpha = \sqrt{2} \). This was continued by Littlewood [17, 18], who considered generalizations to Taylor series whose coefficients have smoothly varying moduli and arguments of the form \( \exp(2\pi i \alpha n^2) \), where \( \alpha \) is a quadratic irrationality.

Such functions behave similarly to random entire functions previously studied by Levy [16] and Littlewood and Offord [19], in particular they display the “pits effect” which Littlewood described as follows:

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"If we erect an ordinate $|f(z)|$ at the point $z$ of the $z$-plane, then the resulting surface is an exponentially rapidly rising bowl, approximately of revolution, with exponentially small pits going down to the bottom. The zeros of $f$, more generally the $w$-points where $f = w$, all lie in the pits for $|z| > R(w)$. Finally the pits are very uniformly distributed in direction, and as uniformly distributed in distance as is compatible with the order $\rho$".

The earliest study of functions (1) known to the authors is the thesis of Alander [1] who considered the case of rational $\alpha$. Levy [16] used the results of Hardy and Littlewood on Diophantine approximation to prove the following. Let

$$M(r, f) = \max_{|z|=r} |f(z)| \quad \text{and} \quad m^2(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta.$$ 

Then

$$M(r, f)/m^2(r, f) \quad \text{is bounded} \quad (2)$$

for $f$ of the form (1), and $\alpha$ satisfying a Diophantine condition. This is even stronger regularity than random arguments of coefficients yield [16, 19]. Some other works where the function (1) with various $\alpha$ was studied or used are [7, 8, 20, 28].

Function (1) is the unique analytic solution of the functional equation

$$f'(z) = qf(q^2z), \quad \text{where} \quad q = e^{2\pi i \alpha}, \quad \text{and} \quad f(0) = 1, \quad (3)$$

which is a special case of the so-called “pantograph equation”. There is a large literature on this equation with real $q$, see, for example, [14, 13] and references there.

Recently there was a renewed interest to the functions of the type (1) because they arise as the limits as $q \to e^{2\pi i \alpha}$ of the function of two variables

$$\sum_{n=0}^{\infty} q^n z^n/n!$$

which plays an important role in graph theory [27] and statistical mechanics [25]. This function is the unique solution of (3), for all $q$ in the closed unit disc.

In the present paper, we first study arbitrary entire functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n/n!, \quad \text{where} \quad |a_n| = 1. \quad (4)$$
Our Theorem 1 says that such functions cannot decrease exponentially on any ray, unless \( f \) is an exponential. This can be compared with a result of Rubel and Stolarski [23] that there exist exactly five series of the form (4) with \( a_0 = 0, \ a_n = \pm 1 \) which are bounded on the negative ray. Our second result, Theorem 2 shows that one cannot replace the condition of exponential decrease in Theorem 1 by boundedness on a ray: there are infinitely many functions of the form (4) which tend to zero as \( z \to \infty \) in the closed right half-plane.

In the second part of the paper, we consider the case \( \arg a_n = 2\pi \sin^2 \alpha \) with any irrational \( \alpha \). Theorem 3 shows that the qualitative picture of \( |f(z)| \) is the same as described by Littlewood, except that our estimate of the size of the pits is worse than exponential. In particular, we show that

\[
\log |f(z)| = |z| + o(|z|),
\]

outside some exceptional set of \( z \). According to the Levin–Pfluger theory [15], this behavior of \( |f| \) has the following consequences about the zeros \( z_k \) of \( f \):

The number \( n(r, \theta_1, \theta_2) \) of zeros (counting multiplicity) in the sector

\[
\{z : \theta_1 < \arg z < \theta_2, \ |z| < r\}
\]
satisfies

\[
n(r, \theta_1, \theta_2) = \frac{\theta_2 - \theta_1}{2\pi} (r + o(r)) \quad \text{as} \quad r \to \infty. \tag{5}
\]

Moreover, the limit

\[
\lim_{R \to \infty} \sum_{|z_k| \leq R} \frac{1}{z_k} \quad \text{exists,} \tag{6}
\]

where \( z_k \) is the sequence of zeros of \( f \). It is easy to see from the Taylor series of \( f \) that this limit equals \(-q\).

Thus the Diophantine conditions used in [16, 28, 21] are unnecessary for the qualitative picture of behavior of \( |f| \), but with arbitrary irrational \( \alpha \) the results are less precise than those where \( \alpha \) satisfies a Diophantine condition. Theorem 4 shows that Levy’s result (2) cannot be extended to arbitrary irrational \( \alpha \). Finally we prove a result similar to Theorem 3 where the condition \( |a_n| = 1 \) is replaced by a more flexible condition on the moduli of the coefficients allowing the function to have any order of growth.
We denote by

$$F(z) = \sum_{n=1}^{\infty} a_{n-1} z^{-n},$$

the Borel transform of $f$ in (4) (terminology of [15]). Then $F$ has an analytic continuation from a neighborhood of infinity to the region $\mathbb{C}\setminus K$, where $K$ is a convex compact set in the plane, which is called the conjugate indicator diagram. The indicator

$$h_f(\theta) := \limsup_{r \to \infty} r^{-1} \log |f(re^{i\theta})|, \quad |\theta| \leq \pi,$$

is the support function of the convex set symmetric to $K$ with respect to the real axis.

We also consider the function

$$G(z) = \sum_{n=1}^{\infty} a_{n-1} z^n$$

analytic in the unit disc. Transition from $F$ to $G$ is by the change of the variable $1/z$.

**Pólya's theorem** ([15, Appendix I, §5]). Suppose that $G$ has an analytic continuation from the unit disc to infinity through some angle $|\arg z - \pi| < \delta$. Then the coefficients $a_n$ can be interpolated by an entire function $g$ of exponential type such that the indicator diagram of $g$ is contained in the horizontal strip $|\Im z| \leq \pi - \delta$. That is $g(n) = a_n$ for $n \geq 1$, and $h_g(\pm \pi/2) \leq \pi - \delta$.

**Carlson’s theorem** ([15, Ch. IV, Intro.]). Suppose that the indicator diagram of an entire function $g$ has width less than $2\pi$ in the direction of the imaginary axis, that is $h_g(\pi/2) + h_g(-\pi/2) < 2\pi$. Then $g$ cannot vanish on the positive integers, unless $g = 0$.

**Theorem 1.** Every entire function $f$ of the form (4) has non-negative indicator, unless $a_n = \text{const} \cdot a^n$ for some $a$ on the unit circle, in which case $f(z) = e^{a z}$.

By Borel’s transform, this is equivalent to

**Theorem 1’**. Let $G$ be as above. Then $G$ cannot have an analytic continuation to infinity through any half-plane containing 0, unless $a_n = \text{const} \cdot a^n$ for some $a$. 

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These two theorems give characterizations of the exponential function and the geometric series, respectively, showing that their behavior is quite exceptional. Another somewhat similar characterization follows from the result in [23] mentioned above. As a corollary from Theorem 1 we obtain the following result of Carlson [4]: If $z_n$ is the sequence of zeros of $f$ as in (4), then
\[ \sum_n \frac{1}{|z_n|} = \infty, \tag{7} \]
unless $f$ is an exponential.

Proof of Theorem 1'. Suppose that $G$ has such an analytic continuation. Replacing $z$ by $az$ with $|a| = 1$ we achieve that $G$ has an analytic continuation to infinity through some left half-plane of the form $\Re z < \epsilon$, where $\epsilon > 0$.

Pólya’s theorem then implies that $a_n = g(n)$ for some entire function whose indicator diagram is contained in the strip $|\Im z| < \pi/2 - \delta$, for some $\delta > 0$. Consider the functions
\[ g_R(z) = (g(z) + \overline{g(z)})/2 \quad \text{and} \quad g_I(z) = (g(z) - \overline{g(z)})/(2i). \]
On the real axis we have $g_R(x) = \Re g(x)$ and $g_I(x) = \Im g(x)$. Consider the entire function
\[ H = g_I^2 + g_R^2. \]
Then at positive integers we have
\[ H(n) = g_I^2(n) + g_R^2(n) = (\Re g(n))^2 + (\Im g(n))^2 = |a_n|^2 = 1. \]
So the function $H - 1$ has zeros at all positive integers. Its indicator diagram is contained in the strip
\[ |\Im z| \leq \pi - 2\delta, \]
(Squaring stretches the indicator diagram by a factor of 2, and the indicator diagram of the sum of two functions is contained in the convex hull of the union of their diagrams). Now, by Carlson’s theorem, $H \equiv 1$, so
\[ g_I^2 + g_R^2 = 1. \tag{8} \]
The general solution of this functional equation in the class of entire functions is $g_I = \cos \circ \phi$, $g_R = \sin \circ \phi$, where $\phi$ is an entire function. It is well-known and easy to see that for $g_I$ and $g_R$ to be of exponential type, it is necessary and sufficient that $\phi(z) = cz + b$. As $g_I$ and $g_R$ are real on the real line,
we conclude that $c$ and $b$ are real. Thus $a_n = \cos(cn + b) + i\sin(cn + b) = \text{const} \cdot e^{icn} = \text{const} \cdot a^n$, as advertised.

An alternative way to derive the conclusion from (8) suggested by Kat-
snelson is to notice that (8) implies

$$|g(x)| \equiv 1 \quad \text{for real } x.$$  \hspace{0.5cm} (9)

The Symmetry Principle then implies that $g$ has no zeros (if $z_0$ is a zero then $\overline{z}_0$ would be a pole). So $g$ is a function of exponential type without zeros, so $g = \exp(icz)$, where $c$ should be real by (9).

As we already noticed, Theorem 1 implies (7). However it does not imply that the sequence of zeros has positive density: there exist functions of exponential type, even with constant indicator, whose zeros have zero density.\footnote{Valiron [28, p. 415] erroneously asserted the contrary: that for functions with constant indicator, zero cannot be a Borel exceptional value.}

To construct such examples, take zeros of the form

$$z_k = \left( e^{i \log \log(k+1)} - e^{i \log \log k} \right)^{-1},$$

and construct the canonical product $W$ of genus one with such zeros. It is not hard to show that the asymptotic behavior of this product will be

$$\log |W(re^{i\theta})| = (cr + o(r)) \cos(\theta - \log \log r), \quad r \to \infty$$

outside of some small exceptional set, so the indicator $h_W$ is constant, while the density of zeros is zero.

There exist entire functions of the form (4), other than the exponential, which are bounded in the left half-plane. The simplest example is Hardy’s generalization of $e^z$ defined by the power series

$$E_{s,a} = \sum_{n=1}^{\infty} \frac{(n+a)^s z^n}{n!}, \quad s \in \mathbb{C}, \quad a > 0.$$

For pure imaginary $s$, this series is of the form (4). Hardy [10] proved the asymptotic formula

$$E_{s,a}(z) = z^s e^z (1 + o(1)) + \frac{\Gamma(a)}{\Gamma(-s)(-z)^a \log(-z)} (1 + o(1)),$$
as $z \to \infty$, $|\arg z \pm \pi/2| < \epsilon$, for every $\epsilon \in (0, \pi/2)$. This formula implies that the functions $E_{s,a}$ with pure imaginary $s$ are bounded in the closed left half-plane. For further results on Hardy’s function, see [22].

**Theorem 2.** Let $\psi$ be a real entire function with the property

$$\psi(\zeta) = o(|\zeta|), \quad \zeta \to \infty$$

in every half-plane $\Re \zeta > c$, $c \in \mathbb{R}$. Then the function

$$f(z) = \sum_{n=0}^{\infty} \frac{e^{i\psi(n)}}{n!} z^n$$

is of the form (4) and for every $A > 0$ and every $\epsilon > 0$ we have

$$|f(re^{i\phi})| = O(r^{-A}), \quad r \to \infty,$$  \hspace{1cm} (10)

uniformly for $|\phi - \pi| \leq \pi/2 - \epsilon$.

**Proof.** We have the following integral representation:

$$f(-z) = -\frac{1}{2\pi i} \int_{-\infty}^{A+i\infty} \frac{\pi e^{i\psi(\zeta)} z^\zeta}{\Gamma(\zeta+1) \sin \pi \zeta} d\zeta = \frac{1}{2\pi i} \int_{-\infty}^{A-i\infty} \frac{e^{i\psi(\zeta)} z^\zeta \Gamma(-\zeta) d\zeta}{\Gamma(\zeta+1) \sin \pi \zeta},$$  \hspace{1cm} (11)

where $A > 0$ is any positive number. To obtain this representation, we notice that that by Stirling’s formula, the modulus of the integrand does not exceed

$$|z|^{-\Re \zeta} \exp \left(\left(-\pi/2 + \phi + o(1)\right) |\Im \zeta|\right),$$

as $|\zeta| \to \infty$ in every half-plane of the form $\Re \zeta \geq -A$. Here $o(1)$ is independent of $z$. Applying the residue formula to the rectangle

$$\{\zeta : -c < \Re \zeta < N + 1/2, |\Im \zeta| < N + 1/2\},$$

and letting $N$ tend to infinity, we obtain (11). Now the same estimate of the integrand shows that (10) holds. $\square$

Theorem 1 implies that the indicator diagram of a function of the form (4), other than an exponential, contains zero. Theorem 2 shows that the indicator diagram of such a function can be contained in a closed half-plane. It seems interesting to describe all possible indicator diagrams that can occur for functions of the form (4). We have the following partial result.
**Proposition.** For arbitrary finite set $Z$ on the unit circle, there exists an entire function of the form (4) whose indicator diagram coincides with the convex hull of $Z \cup -Z$.

**Proof.** Let $E$ be the set of all entire functions of the form (4). We consider the following operators on $E$:

$$R_\theta[f](z) := f(ze^{-i\theta}),$$

$$C[f](z) := \frac{1}{2} \left( f(z) + f(-z) \right),$$

and

$$S[f](z) := \frac{1}{2} \left( f(z) - f(-z) \right).$$

Now we define an operator $E \times E \to E$ by the formula

$$Q_{\theta_1, \theta_2}[f_1, f_2] = (C \circ R_{\theta_1})[f_1] + (S \circ R_{\theta_2})[f_2].$$

It can be easily shown that if $f \in E$ is a function with indicator diagram $[0,1]$, then $(C \circ R_{\theta})[f]$ and $(S \circ R_{\theta})[f]$ have indicator diagram $[-e^{i\theta}, e^{i\theta}]$. Hence the indicator diagram of $f_1 = Q_{\theta_1, \theta_2}[f, f]$ is the convex hull of

$$\{ e^{i\theta_1}, -e^{i\theta_1}, e^{i\theta_2}, -e^{i\theta_2} \}.$$ 

This proves the Proposition for the sets $Z$ of two points. Then we consider $f_2 = Q_{0, \theta_2}[f_1, f]$ and so on. \qed

Now we consider functions of the form (4) with $\arg a_n = 2\pi n^2 \alpha$, $\alpha \in \mathbb{R}$.

**Theorem 3.** Let $f$ be of the form (4) with $a_n = \exp(2\pi in^2 \alpha)$, where $\alpha$ is irrational. Then $f$ has completely regular growth in the sense of Levin–Pfluger, and $h_f \equiv 1$.

We recall the main facts of the Levin–Pfluger theory in the modern language [2]. Fix a positive number $\rho$. Let $u$ be a subharmonic function in the plane satisfying

$$u(z) \leq O(r^\rho), \quad r \to \infty.$$ 

Then the family of subharmonic functions

$$A_t u(z) = t^{-\rho} u(tz), \quad t > 1,$$
is bounded from above on every compact subset of the plane. Such families of subharmonic functions are pre-compact in the topology $D'$ of Schwartz’s distributions [11, Theorem 4.1.9], so from every sequence $A_{t_k}u$, $t_k \to \infty$ one can select a convergent subsequence. An entire function $f$ or order $\rho$, normal type is called of completely regular growth if the limit

$$u = \lim_{t \to \infty} A_t \log |f|$$

exists. It is easy to see that this limit is a fixed point for all operators $A_t$, so it has the form

$$u(re^{i\theta}) = r^\rho h(\theta),$$

and $h$ is the indicator of $f$. Operators $A_t$ also act on measures in the plane by the formula

$$A_t \mu(E) = t^{-\rho} \mu(tE) \quad \text{for} \quad E \subset \mathbb{C}.$$  

The Riesz measure $\mu_f$ of $\log |f|$ is the counting measure of zeros of $f$, and one of the results of Levin–Pfluger can be stated as follows: The existence of the limit (12) implies the existence of the limit

$$\mu = \lim_{t \to \infty} A_t \mu_f.$$  

This limit $\mu$ is also fixed by all operators $A_t$, so

$$d\mu = r^{\rho - 1}drd\nu(\theta),$$

where $\nu$ is a measure on the unit circle which is called the angular density of zeros. This measure $\nu$ is related to the indicator by the formula

$$d\nu = (h'' + \rho^2h)d\theta,$$

in the sense of distributions.

Thus, as a corollary from Theorem 3, we obtain that the angular density of zeros of $f$ is a constant multiple of the Lebesgue measure.

Completely regular growth with indicator 1 and order $\rho = 1$ implies that

$$\log |f(re^{i\theta})| = r + o(r) \quad \text{as} \quad r \to \infty,$$

uniformly with respect to $\theta$, when $re^{i\theta}$ does not belong to an exceptional set. According to Azarin, [2], for every $\eta > 0$, this exceptional set can be covered by discs centered at $w_k$ and of radii $r_k$ such that

$$\sum_{|w_k| \leq r} r_k^\eta = o(r^\eta), \quad r \to \infty.$$
This improves the original condition with $\eta = 1$ given in [15]. The properties (5) and (6) of zeros of $f$, stated in the beginning of the paper, follow from (13) by theorems II.2 and III.4 in [15], see also [24].

The exceptional set (14) is larger than the exceptional set in the work of Nassif. The exceptional set in Theorem 3 could be improved to a set of exponentially small circles if one knew that the zeros of $f$ are well separated. This seems to be an interesting unsolved problem about the function (1). In particular, can $f$ of the form (1) have a multiple zero? For $\alpha = \sqrt{2}$, Nassif proved that all but finitely many zeros are simple and well separated.

That the indicator of $f$ in Theorem 3 is constant was proved by Valiron [28, p. 412]. This also follows from the result of Cooper [6], who proved that the corresponding function $G$ has the unit circle as its natural boundary, see also [5, p. 76, Footnote] where a short proof of Cooper’s theorem is given. However, as we noticed above, constancy of the indicator by itself only implies (7); it is the statement about completely regular growth that permits to conclude that the zeros have positive density.

Proof of Theorem 3. By differentiating the power series it is easy to obtain

$$f'(z) = e^{2\pi i \alpha} f(ze^{i\beta}), \quad \text{where } \beta = 4\pi \alpha.$$  \hfill (15)

(This is the “pantograph equation” (3) with $q = e^{2\pi i \alpha}$.) The assumption that $|a_n| = 1$ implies the following behavior of $M(r, f)$

$$\log M(r, f) = r + o(r).$$  \hfill (16)

This is proved by the standard argument relating the growth of $M(r, f)$ with the moduli of the coefficients, see, for example [15, Ch. 1, §2]. The bounds $0 \leq r - \log M(r, f) \leq (1/4 + o(1)) \log r$ can be obtained as follows. The upper bound $M(r, f) \leq e^r$ is trivial, and for the lower bound, use Cauchy’s inequality $M(r, f) \geq r^n/n!$, and maximize the right hand side with respect to $n$. In particular, the order $\rho = 1$.

It follows from (16) that the family of subharmonic functions

$$\{ u_t = A_t \log |f| : 0 < t < \infty \},$$

Valiron obtained the equation which is equivalent to our (20) below, [28, Eq. (11)] but he did not fully explore its consequences. Later in the same paper, on p. 421, Valiron proves that $f$ is of completely regular growth only under an additional Diophantine condition on $\alpha$. 

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is uniformly bounded from above on compact subsets of $\mathbf{C}$. Moreover, $u_t(0) = 0$. So every sequence $\sigma = (t_k) \to \infty$ contains a subsequence $\sigma'$ such that the limit
\begin{equation}
    u = \lim_{t \in \sigma', t \to \infty} u_t
\end{equation}
exists in $D'$, the space of Schwartz’s distributions in the plane. The set of all possible limits $u$ for all sequences $\sigma$ is called the limit set of $f$ and denoted by $\text{Fr} [f]$. It consists of subharmonic functions in the plane satisfying $u(0) = 0$. Equation, (16) implies that
\begin{equation}
    \max_{|z| \leq r} u(z) = r, \quad 0 \leq r < \infty.
\end{equation}
If $u = \lim t_k^{-1} \log |f(t_k z)|$, and $v = \lim t_k^{-1} \log |f'(t_k z)|$ with the same sequence $t_k \to \infty$, then
\begin{equation}
    v \leq u.
\end{equation}
Indeed, by Cauchy’s inequality, for every $\epsilon > 0$ and $|z| > 1/\epsilon$, we have
\begin{equation*}
    \log |f'(z)| \leq \max_{|\zeta| \leq |z|} \log |f(z + \zeta)|.
\end{equation*}
This implies that for every $\epsilon > 0$,
\begin{equation*}
    v(z) \leq \max_{|\zeta| \leq \epsilon} u(z + \zeta).
\end{equation*}
Now the upper semi-continuity of subharmonic functions shows that the right hand side of the last equation tends to $u(z)$ as $\epsilon \to 0$, which proves (19).

The functional equation (15) and (19) imply that $u(ze^{i\beta}) \leq u(z)$, and this gives
\begin{equation}
    u(ze^{i\beta}) \equiv u(z).
\end{equation}
As $\beta$ is irrational, $u(z)$ is independent of $\arg z$, and taking (18) into account we conclude that the limit set $\text{Fr} [f]$ consists of the single function $u(z) = |z|$. This means that $f$ is of completely regular growth with constant indicator.

Now we show that there exist irrational $\alpha$ such that the corresponding functions $f_\alpha$ in (1) do not have property (2).

**Theorem 4.** There is a residual set $E$ on the unit circle, such that for a function $f_\alpha$ as in (1) with $\alpha \in E$, we have
\begin{equation}
    \limsup_{r \to \infty} M(r, f)/m_2(r, f) = \infty.
\end{equation}
We recall that a set is called residual if it is an intersection of countably many dense open sets. By Baire’s Category Theorem, residual sets on $[0,1]$ have the power of a continuum and thus contain irrational points.

Proof of Theorem 4. Consider the sets

$$E_{m,n} = \{ \alpha : M(r, f_\alpha)/m_2(r, f_\alpha) \leq m \text{ for } r \geq n \},$$

where $m$ and $n$ are positive integers. Evidently, all these sets are closed. Let $E = [0,1] \setminus \cup_{m,n} E_{m,n}$. Then for $\alpha \in E$ we have (21), and $E$ is a countable intersection of open sets. It remains to show that $E$ is dense. We will show that $E$ contains all rational numbers. Indeed, for rational $\alpha$, $f_\alpha$ is a finite trigonometric sum:

$$f_\alpha = \sum c_k e^{b_k z},$$

where $b_k$ are roots of unity. This representation immediately follows from the functional equation (15): iterating this functional equation finitely many times, we obtain a linear differential equation whose solutions are trigonometric sums (22). It is clear that any finite trigonometric sum $g$ satisfies

$$M(r, g)/m_2(r, g) \to \infty \text{ as } r \to \infty.$$

This proves that $E$ is dense and thus residual.

Now we extend Theorem 3 to the case that $|a_n|$ is non-constant. For this we need

Hadamard’s Multiplication Theorem [3]. Let $f = \sum_{n=0}^{\infty} c_n z^n$ be an entire function, and $H = \sum_{n=0}^{\infty} b_n z^n$ a function analytic in $\mathbb{C}\{1\}$. Then the function

$$(f \ast H)(z) = \sum_{n=0}^{\infty} a_n c_n z^n$$

has the integral representation

$$(f \ast H)(z) = \frac{1}{2\pi i} \int_C f(\zeta) H \left( \frac{z}{\zeta} \right) \frac{d\zeta}{\zeta},$$

where $C$ is any closed contour going once in positive direction around the point 1.
This operation \( f \star H \) is called the Hadamard composition of power series. From the integral representation we immediately obtain

\[
| (f \star H)(z) | \leq K \max_{|\zeta - z| \leq r |z|} |f(\zeta)|, \quad \text{where} \quad K = \max_{|\zeta - 1| = r/(1-r)} |H(\zeta)|. \quad (23)
\]

**Theorem 5.** Let \( h \) be an entire function of minimal exponential type. Let\n
\[
c_n = h(0)h(1) \ldots h(n), \quad n \geq 0,
\]

and assume in addition that\n
\[
- \log |c_n| = \frac{1}{\rho} n \log n - cn + o(n), \quad n \to \infty,
\]

with some real constant \( c \). Then the entire function

\[
f(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i n^2 \alpha} z^n,
\]

with irrational \( \alpha \), has order \( \rho \), normal type and completely regular growth with constant indicator.

The condition that \( c_n/c_{n-1} \) is interpolated by an entire function of minimal exponential type is not so rigid as it may seem. In this connection, we recall a theorem of Keldysh (see, for example, [9]) that every function \( h_1 \) analytic in the sector \( |\arg z| < \pi - \epsilon \) and satisfying \( \log |h_1(z)| = O(|z|^\lambda) \), \( z \to \infty \) there, with \( \lambda = \pi/(\pi + \epsilon) < 1 \), can be approximated by an entire function \( h \) of normal type, order \( \lambda \) so that

\[
|h(z) - h_1(z)| = O(e^{-|z|^{\lambda}}), \quad z \to \infty, \quad |\arg z| < \pi - 2\epsilon.
\]

For example, one can take

\[
h_1(z) = z^{-1/\rho}, \quad \rho > 0,
\]

and apply Keldysh’s theorem, to obtain a function \( f \) of normal type, order \( \rho \), satisfying all conditions of Theorem 5.

**Proof of Theorem 5.** We write:

\[
f(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i n^2 \alpha} z^n
\]
\[ 1 + \sum_{n=0}^{\infty} c_{n+1} e^{2\pi i(n+1)^2 \alpha} z^{n+1} \]

\[ = 1 + \sum_{n=0}^{\infty} c_{n+1} e^{2\pi i n^2 \alpha} (ze^{4\pi i n\alpha})^n \]

\[ = 1 + z e^{2\pi i \alpha} (f \ast H)(ze^{i\beta}), \quad \text{where} \quad \beta = 4\pi \alpha, \]

and

\[ H(z) = \sum_{n=0}^{\infty} \frac{c_{n+1}}{c_n} z^n. \]

By the assumption (24) of the theorem, and Pólya’s theorem above, \( H \) is holomorphic in \( \mathbb{C} \setminus \{1\} \), so the estimate (23) holds. Assumption (25) implies that

\[ M(r, f) = \sigma r^\rho + o(r^\rho), \quad r \to \infty, \]

where \( \sigma = e^{e\rho}/(e\rho) \), see [15, Ch. I, §2]. So from every sequence one can select a subsequence such that the limits

\[ u(z) = \lim_{k \to \infty} t_k^{-\rho} \log |f(t_k z)| \quad \text{and} \quad v(z) = \lim_{k \to \infty} t_k^{-\rho} \log |(f \ast H)(t_k z)| \]

exist, and (23) implies that \( v \leq u \), by a similar argument as (19) was derived. Now the equation

\[ f(z) = 1 + z e^{2\pi i \alpha} (f \ast H)(ze^{i\beta}) \]

implies

\[ u(z) \leq \max\{0, v(ze^{i\beta})\} \leq \max\{0, u(ze^{i\beta})\}, \]

and as \( \beta \) is irrational, we conclude that \( u \) does not depend on \( \arg z \). This completes the proof. \( \square \)

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