

Circular pentagons and real solutions of Painlevé VI equations

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Abstract

We study real solutions of a class of Painlevé VI equations. To each such solution we associate a geometric object, a one-parametric family of circular pentagons. We describe an algorithm which permits to compute the numbers of zeros, poles, 1-points and fixed points of the solution on the interval $(1, +\infty)$ and their mutual position. The monodromy of the associated linear equation and parameters of the Painlevé VI equation are easily recovered from the family of pentagons.

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0. Introduction

Consider a linear differential equation

$$w'' - Pw' + Qw = 0, \tag{1}$$

where P and Q are rational functions of the complex independent variable, and assume that all singularities are regular, and all parameters (singular points, exponents and accessory parameters) are real. Then the ratio $f = w_1/w_2$ of two linearly independent solutions maps the upper half-plane onto a

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circular polygon (see Section 3 below for a precise definition). Every simply connected circular polygon can be obtained this way. Klein [29] and Van Vleck [42] used this connection between differential equations with three singularities and circular triangles to count zeros and poles of hypergeometric functions. We use a similar idea to count special points on a real interval of real solutions of Painlevé VI equations

$$q_{xx} = \frac{1}{2} \left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-x} \right) q_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{q-x} \right) q_x \quad (2)$$

$$+ \frac{q(q-1)(q-x)}{2x^2(x-1)^2} \left\{ \kappa_4^2 - \kappa_1^2 \frac{x}{q^2} + \kappa_2^2 \frac{x-1}{(q-1)^2} + (1 - \kappa_3^2) \frac{x(x-1)}{(q-x)^2} \right\},$$

where q is a function of x , and κ_j are real parameters. By definition, *special points* of a solution are those points x for which $q(x) \in \{0, 1, x, \infty\}$, the points where the assumptions of the existence and uniqueness theorem of Cauchy are violated.

Equation (2), which we call PVI, was originally discovered by Picard [38], Painlevé [37] and Gambier [17] as the most general equation of the form

$$q_{xx} = R(q_x, q, x), \quad (3)$$

where R is a rational function of q_x, q whose coefficients are analytic in x , and whose solutions have no movable singularities (poles are not counted as singularities). For PVI, this means that all solutions admit a meromorphic continuation in the region $\mathbf{C} \setminus \{0, 1\}$. Painlevé and Gambier classified all equations (3) without movable singularities, and found that all except six of them can be reduced to linear or first order differential equations. Of the six remaining equations, PVI is the most general, in the sense that the other five can be obtained from it by certain degeneration process.

Meanwhile, Richard Fuchs [16] independently discovered (2) as the condition of isomonodromic deformation of a linear differential equation (1) with five regular singularities, one of them apparent with exponents 0 and 2. Later it was found that Painlevé equations arise in a variety of problems of mathematics and physics [8, 22, 30, 32], and their solutions, called Painlevé transcendents, gradually gain the status of special functions.

Besides applications, most work on PVI falls into three categories: a) transformations of the equation [35, 26], b) search for special solutions, like algebraic ones [31] or those expressed in terms of classical special functions [22], and c) asymptotics at the fixed singularities $0, 1, \infty$, [28, 21].

In this paper, we study real solutions of PVI with real parameters κ_j on a real interval, one of the three intervals between the fixed singularities $(0, 1, \infty)$. Our main result is a combinatorial algorithm (which can be performed without a computer) which determines the number of special points and their mutual position on the interval. The outcome of the algorithm is a sequence of symbols $0, 1, x, \infty$ which shows in which order the solution $q(x)$ of (2) takes these four values as x increases from 1 to ∞ . In particular, we describe those solutions which have no special points on $(1, \infty)$.

So we obtain global, exact (non-asymptotic) results describing qualitative features of a quite general class of solutions, namely real solutions.

We do this by exploring the connection with a linear differential equation of the type (1) with 5 regular singularities discovered by Fuchs. When all parameters of this linear equation are real, the ratio f of any two linearly independent solutions of (1) maps the upper half-plane onto a circular pentagon of a special kind: it is a circular quadrilateral with a slit or a circular triangle with a slit. If the f -preimage of the tip of the slit is not counted as a vertex, our special pentagon can be considered as a conformal quadrilateral. A real solution of PVI describes the relation between the modulus of this conformal quadrilateral and the f -preimage of the tip of the slit. For some values of the modulus the slit vanishes or becomes a cross cut, and the pentagon becomes a circular quadrilateral. These values of the modulus correspond to special points of the PVI solution. Thus the study of the number and mutual position of these special points is equivalent to a geometric problem of describing the evolution of a one-parameter family of special pentagons. We do this using a representation of circular polygons by cell decompositions of a disk which are called nets. This method was developed in [9, 12, 10, 13, 14] for other problems.

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1. A class of linear differential equations

We consider the class of linear differential equations (1) of the form

$$w'' - \left(\frac{1}{z-q} + \sum_{j=1}^3 \frac{\kappa_j - 1}{z-t_j} \right) w' + \left(\frac{p}{z-q} - \sum_{j=1}^3 \frac{h_j}{z-t_j} \right) w = 0, \quad (4)$$

where

$$(t_1, t_2, t_3) = (0, 1, x), \quad (5)$$

$x > 1$, $\kappa_j \geq 0$, p and q are real numbers. We impose the following conditions:

- a) ∞ is a regular singularity, with exponent difference $\kappa_4 \geq 0$,
- b) q is an apparent singularity, but the singularities at $(0, 1, x, \infty)$ are not apparent (have non-trivial local projective monodromy).

It follows from the form of (4) that the exponents at q are 0 and 2, and t_j are regular singularities with exponents 0 and κ_j for $1 \leq j \leq 3$. The exponents at ∞ are determined from the Fuchs relation and from the assumption that their difference is κ_4 .

Conditions a), b) determine the h_j uniquely in terms of p, q, x and κ_j , $1 \leq j \leq 4$, by solving the following system of linear equations:

$$\sum_{j=1}^3 h_j = p, \quad (6)$$

$$\sum_{j=1}^3 t_j h_j = pq + \frac{\kappa_4^2 - 1}{4} - \frac{1}{4} \left(\sum_{j=1}^3 \kappa_j \right)^2 + \frac{1}{2} \sum_{j=1}^3 \kappa_j, \quad (7)$$

$$\sum_{j=1}^3 \frac{h_j}{q - t_j} = p^2 - p \sum_{j=1}^3 \frac{\kappa_j - 1}{q - t_j}. \quad (8)$$

Equations (6) and (7) correspond to condition a), while equation (8) corresponds to condition b).

The determinant of this system is

$$\frac{x(x-1)}{q(q-1)(q-x)},$$

thus for given real $\kappa_j \geq 0$, p , $x \notin \{0, 1\}$, and $q \notin \{0, 1, x\}$ the coefficients h_j are uniquely defined real numbers. Our notation in (2) and (4) is the same as in [26], but we notice a misprint in the first line of [26, (2.1)]: q must be q_x .

2. Isomonodromic deformation and PVI

Let us choose the generators of the fundamental group of $\Omega = \mathbf{C} \setminus \{0, 1, x\}$ as shown in Fig. 1.

Let (w_1, w_2) be a pair of linearly independent solutions of (4) normalized by

$$\begin{pmatrix} w_1(x_0) & w_2(x_0) \\ w_1'(x_0) & w_2'(x_0) \end{pmatrix} = I. \quad (9)$$

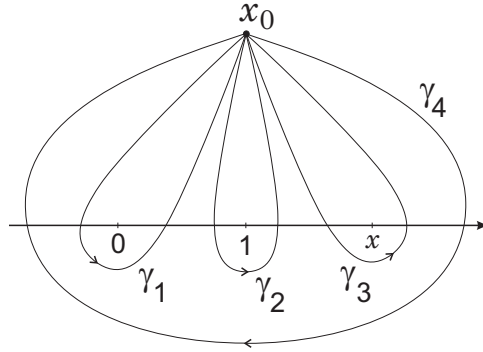


Figure 1: Loops defining M_1 , M_2 , M_3 , and M_4 .

Performing an analytic continuation of these solutions along an element $\gamma \in \pi_1(\Omega, x_0)$ we obtain

$$(w_1^\gamma, w_2^\gamma) = (w_1, w_2)M_\gamma$$

for some $M_\gamma \in GL(2, \mathbf{C})$. Notice that the map $\gamma \mapsto M_\gamma$ is an anti-representation of the fundamental group¹

For the ratio $f = w_1/w_2$ we obtain

$$f^\gamma = T_\gamma \circ f,$$

where T_γ is a linear-fractional transformation. We identify the group of linear-fractional transformations with $PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\{\pm I\}$, and the quadruple (T_1, T_2, T_3, T_4) is the set of generators of the *projective monodromy representation* $\pi_1(\Omega, x_0) \rightarrow PSL(2, \mathbf{C})$. The correspondence $\gamma \rightarrow T_\gamma$ is a group homomorphism. The generators are chosen so that

$$T_1 T_2 T_3 T_4 = \text{id}, \tag{10}$$

and we assume that

$$T_j \neq \text{id}, \quad 1 \leq j \leq 4. \tag{11}$$

For the matrices representing the generators T_j we use the same letters, and they are related to the matrices M_j by

$$T_j = \frac{1}{\sqrt{\det M_j}} M_j^T, \quad 1 \leq j \leq 4,$$

¹For the fundamental group we use the standard notation: product $\gamma_1 \gamma_2$ means that the path γ_1 is followed by the path γ_2 .

where T stands for the transposition.

When the parameters κ_j are fixed, the projective monodromy representation of equation (1) depends on x , p and q .

When we change x and deform the loops continuously, the condition that the monodromy matrices do not change is that $p = p(x)$ and $q = q(x)$ satisfy the following non-autonomous Hamiltonian system [26, (3.7)]:

$$\frac{dq}{dx} = \frac{\partial h}{\partial p}, \quad \frac{dp}{dx} = -\frac{\partial h}{\partial q}.$$

Here the Hamiltonian $h = h_3$ is given by²

$$\begin{aligned} x(x-1)h &= q(q-1)(q-x)p^2 \\ &- \{(\kappa_3-1)q(q-1) + \kappa_1(q-1)(q-x) + \kappa_2q(q-x)\}p \\ &+ \kappa_0(\kappa_0 + \kappa_4)(q-x), \end{aligned}$$

where

$$\kappa_0 = (1 - \kappa_1 - \kappa_2 - \kappa_3 - \kappa_4)/2.$$

This Hamiltonian system is equivalent to (2). All solutions of PVI are obtained in this way.

Special points of $q(x)$ correspond to collisions of the singular point q with one of the four other singular points of equation (4). Thus when x is a special point, (1) becomes an equation with four regular singularities (Heun's equation).

In this paper we consider parameters $\kappa_j \geq 0$, $1 \leq j \leq 4$, and real solutions $q(x)$ of (2) defined for $x \in (1, +\infty)$. In view of the formulas (6), (7), (8), in this case all parameters in (4) are real.

The condition on the monodromy matrices that ensures that the solution of PVI is real is discussed in Appendix I.

Remark. A more general class of real equations (2) is obtained by allowing some κ_j be pure imaginary. In this case, equation (4) also has a geometric interpretation [39], but very different from the interpretation in this paper: the developing map f (defined in the next section) still maps the upper half-plane onto a Riemann surface bounded by four circles, but when some κ_j are imaginary, this Riemann surface has infinitely many sheets.

3. Circular polygons

²See (4) for the definition of h_3 .

A *circular n -gon* is a bordered surface homeomorphic to a closed disk, spread over the sphere without ramification points in the interior, and such that the border consists of n arcs and n points separating them, so that each arc projects into a circle on the sphere locally injectively.

To give a more formal definition, we denote by S the conformal sphere (the unique compact simply connected Riemann surface). A *circle* in S can be defined by using only the conformal structure: it is the set of fixed points of an anti-conformal involution. Conformal automorphisms of S send circles to circles.

Let \overline{D} be a conformal closed disk³, and let $\{t_j\}$ be n distinct boundary points enumerated according to the standard orientation of ∂D . In what follows we understand the subscript j in t_j and in other similar notations as a residue modulo n .

A *developing map* is a continuous function $f : \overline{D} \rightarrow S$ which is holomorphic in $\overline{D} \setminus \{t_1, \dots, t_n\}$ and satisfies

$$f'(z) \neq 0, \quad z \in \overline{D} \setminus \{t_1, \dots, t_n\}, \quad (12)$$

$$f(z) \sim f(t_j) + c_j(z - t_j)^{\alpha_j} \quad \text{as } z \rightarrow t_j, \quad (13)$$

where $\alpha_j > 0$ and $c_j \in \mathbf{C}^*$, or

$$f(z) \sim f(t_j) + c_j / \log(z - t_j) \quad \text{as } z \rightarrow t_j, \quad (14)$$

and such that $f([t_{j-1}, t_j])$ are contained in some circles $C_j \subset S$. (These formulas need an evident modification if $f(t_j) = \infty$, or $f(z) = \infty$ in (12).) The circles C_j need not be distinct. A *circular n -gon* is identified with the ordered set

$$(\overline{D}, t_1, \dots, t_n, f). \quad (15)$$

Sometimes we will omit the word “circular”, calling these objects simply polygons (digons, triangles, quadrilaterals, etc.)

Two circular polygons

$$(\overline{D}_1, t'_1, \dots, t'_n, f_1) \quad \text{and} \quad (\overline{D}_2, t''_1, \dots, t''_n, f_2) \quad (16)$$

are considered *equal* if there exists a conformal map $\phi : \overline{D}_1 \rightarrow \overline{D}_2$ such that $\phi(t'_j) = t''_j$ and $f_1 = f_2 \circ \phi$. If the last equality is replaced by $f_1 = L \circ f_2 \circ \phi$, where $L \in \text{Aut } S$ then the two polygons are called *equivalent*.

³The closure of a Jordan region.

The points t_j are called *corners* and the arcs (t_{j-1}, t_j) *sides* of a polygon. The angle at t_j is defined as α_j in (13), and we set $\alpha_j = 0$ if (14) holds.

Notice that we measure the angles in half-turns rather than radians.

We denote by C_j the circle containing $f([t_{j-1}, t_j])$. Then we obtain n labeled circles with the property that

$$C_j \cap C_{j+1} \neq \emptyset, \quad j \in \mathbf{Z}_4. \quad (17)$$

Indeed, $f(t_j)$ belongs to this intersection. Any such sequence of circles will be called an *n-circle chain*, or simply a chain when it is clear what n is.

Notice that 0-gons are just disks, while 1-gons are disks with one marked point where the angle is 1.

Sometimes it will be convenient to use a Riemannian metric on our polygons. To introduce it, start with the standard spherical Riemannian metric of curvature 1 on S and pull it back to \overline{D} via f . The resulting metric ρ on \overline{D} is a conformal Riemannian metric of curvature 1 on $\overline{D} \setminus \{t_1, \dots, t_n\}$, has conic singularities with the angles α_j at t_j , and each side (t_j, t_{j+1}) has constant geodesic curvature. All metric spaces with these properties arise from circular polygons.

In what follows the word “distance” will always mean *intrinsic distance*: the infimum of lengths of curves connecting two points, where the length of a curve is measured using the intrinsic metric. The area of an n -gon is also measured in the pull-back spherical metric. It is easy to see that all our polygons have finite areas, moreover, the preimages of points under developing maps are finite.

Polygons are equal if and only if the corresponding metric spaces are isometric. Of course, equivalent polygons may be different as metric spaces⁴.

Gluing of two polygons.

We will use the operation of *gluing* circular polygons along a “matching” boundary arc. Suppose that for two polygons in (16), D_1 and D_2 are the upper and lower halves of the unit disk, the interval $(-1, 1)$ contains no corners of either polygon, and is mapped by f_1 and f_2 to the same arc of a circle⁵ (This “arc” can be longer than the whole circle.) Then there exist

⁴One could use only $PSL(2, \mathbf{C})$ -invariant notions, like cross-ratios instead of distances etc., as Klein does. But we find the metric notions more convenient and more intuitive.

⁵This means that there is an increasing diffeomorphism $\psi : [-1, 1] \rightarrow [-1, 1]$ such that $f_2(t) = f_1 \circ \psi(t)$, $t \in [-1, 1]$.

a simple curve γ in the unit disk D with endpoints at ± 1 dividing D into two regions D' and D'' and conformal homeomorphisms $\phi_1 : D' \rightarrow D_1$ and $\phi_2 : D'' \rightarrow D_2$ such that $\phi_j(\pm 1) = \pm 1$ and $f_1 \circ \phi_1(z) = f_2 \circ \phi_2(z)$, $z \in [0, 1]$. Such conformal homeomorphisms ϕ_j exist by a theorem of Lavrentiev, [19, Ch. VI, §1]. Then

$$f(z) = \begin{cases} f_1 \circ \phi_1(z), & z \in D' \\ f_2 \circ \phi_2(z), & z \in D'' \end{cases}$$

extended by continuity on γ , is the developing map of a new polygon which is called the gluing of our two polygons along the common boundary arc.

Variation of the slit.

Consider an $(n+1)$ -gon $Q = (\overline{D}, t_1, \dots, t_n; q, f)$, where the corner q can be anywhere between the t_j , this is why we use a different notation for this corner. Suppose that the angle at q equals 2 and the f -images of the two sides meeting at q belong to the same circle C . (If $q \in (t_{k-1}, t_k)$ then $C = C_k$.) This means that f maps a small neighborhood V of q in D homeomorphically onto a disk centered at $f(q)$ with a slit from the center to the circumference along an arc of the circle C .

In this situation we say that the polygon has a slit, and $b := f(q)$ is called the *tip of the slit*. The slit itself is formally defined as follows:

The slit is the maximal interval $[t, t']$ such that $q \in [t, t'] \subset [t_{k-1}, t_k]$ and the intrinsic lengths of $[t, q]$ and $[q, t']$ are equal.

Consider the small arc $\gamma \in D$ with endpoints q and $c \in D$ which is defined by $\gamma = f^{-1}(C) \cap V$.

Let ϕ be a conformal map of D onto $D \setminus \gamma$. Then $f_1 = f \circ \phi$ defines a new $(n+1)$ -gon with corners $t'_j = \phi^{-1}(t_j)$ for $1 \leq j \leq n$ and $q' := \phi^{-1}(c)$. We say that this new polygon is obtained from the old one by *lengthening the slit*, and the old polygon is obtained from the new one by *shortening the slit*.

Here is an alternative explanation of variation of the slit. Suppose that $D = H$ and $q \in (t_{k-1}, t_k)$. As the sides (t_{k-1}, q) and (q, t_k) are mapped by f to the same circle C , we can extend f by reflection to the lower half-plane H^* . The resulting function \tilde{f} is meromorphic in $G = H \cup H^* \cup (t_{k-1}, t_k)$, maps (t_{k-1}, t_k) into a circle C and has exactly one simple critical point at q . Let σ be the reflection in C . Choose a small disk B centered at $f(q)$ and let $V \subset G$ be the component of $\tilde{f}^{-1}(B)$ which contains q . Let ψ_s be a family of diffeomorphisms of the sphere S , which commutes with σ , whose restriction to $S \setminus B$ is the identity map, and which moves $\tilde{f}(q)$ to a point $s \in C$ near $f(q)$.

Then the main existence theorem for quasiconformal mappings in [2] implies that there is a quasiconformal homeomorphism $\phi_s : G \rightarrow G$ which commutes with complex conjugation and such that $f_s = \psi_s \circ f \circ \phi_s$ is holomorphic. The restriction of f_s onto H is the developing map of the deformed polygon $Q' = (\overline{H}, t'_1, \dots, t'_4; q', f_s)$ where $t'_j = \phi_s^{-1}(t_j)$ and $q' = \phi_s^{-1}(q)$. The dependence of f_s on s is real analytic.

Whenever we have a slit it can be lengthened or shortened. This operation does not affect the angles, the chain of the polygon, or the images of the sides other than those two meeting at q .

4. Relation between equation (4) and a class of circular pentagons

We consider equation (4) satisfying conditions a) and b) after (4).

Proposition 1. *If $p \in \mathbf{R}$, $x > 1$, and $q \in \mathbf{R} \setminus \{0, 1, x\}$, then the ratio f of any two linearly independent solutions of (4) is the developing map of a circular pentagon with $D = H$, and corners at $(t_1, t_2, t_3, t_4) = (0, 1, x, \infty)$ and q , with the angles κ_j at t_j , $1 \leq j \leq 4$, and 2 at q . The f -images of the two sides meeting at q belong to the same circle.*

Conversely, the developing map of every circular pentagon with such properties is the ratio of two solutions of an equation (4) satisfying conditions a), b), with all parameters real, and $0, 1, x, q$ all distinct.

The projective monodromy group of (4) consists of the products of even numbers of reflections in the sides of the pentagon. Condition (11) holds if and only if no pair of sides meeting at t_j , $1 \leq j \leq 4$, is mapped by the developing map into the same circle.

Notice that q can be on any of the four intervals (t_{j-1}, t_j) , $j \in \mathbf{Z}_4$.

Proof. Let $f = w_1/w_2$ be the ratio of linearly independent solutions. Then $f' = (w'_1 w_2 - w_1 w'_2)/w_2^2$, so f is locally univalent in the upper half-plane. If we impose real initial conditions at some real non-singular point, both solutions will be real, and f will be real on the interval between the singularities containing this point. Any other initial condition will give new f related to the old one by a linear-fractional transformation, so $f(z)$ maps every interval between the singular points onto an arc of a circle. The exponents at a singular point t_j are 0 and κ_j if $1 \leq j \leq 3$, so locally $f(z)$ behaves as in (13), (14). At the point q , the exponents are 0 and 2, so the angle is 2, and this point is a removable singularity of f by condition b) after (4), so the sides

meeting at q are mapped to the same circle.

For the converse statement, suppose that a circular pentagon with $D = H$ is given, with the angles $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ at $(0, 1, x, \infty)$ and 2 at q , such that the sides meeting at q are mapped by f into the same circle. Then f extends by reflections to the universal cover of $\mathbf{C} \setminus \{0, x, 1, \infty\}$, and the monodromy of the extended map is a subgroup of $PSL(2, \mathbf{C})$, the group of linear-fractional transformations. This means that the Schwarzian derivative

$$R := \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \quad (18)$$

is single valued, and the local behavior at t_j , ∞ and q implies that R has poles of order two with

$$R(z) = \frac{1 - \kappa_j^2}{2(z - t_j)^2} + \dots, \quad \text{as } z \rightarrow t_j,$$

$$R(z) = -\frac{3}{2(z - q)^2} + \dots, \quad \text{as } z \rightarrow q,$$

and similarly at infinity, so R is a rational function. As the intervals of the real line between the singularities are mapped to arcs of circles, R is real on the real line. As t_j, q and κ_j are real, we conclude that the residues of R are also real. Then the general solution of the Schwarz differential equation (18) is a ratio of two linearly independent solutions of (4), see, for example [18], [24]. The condition that the images of the sides meeting at q belong to the same circle ensures that f has trivial monodromy at q , so q is an apparent singularity with exponents 0 and 2. This completes the proof of Proposition 1.

If C_k is the circle containing $f([t_{k-1}, t_k])$ then (C_1, C_2, C_3, C_4) is a chain of *four* circles for which (17) holds and

$$C_j \neq C_{j+1}, \quad j \in \mathbf{Z}_4 \quad (19)$$

in view of the condition (11). If we denote by σ_j the reflection in C_j , then the projective monodromy generators are

$$T_j = \sigma_j \sigma_{j+1}, \quad j \in \mathbf{Z}_4. \quad (20)$$

This assumes that the fundamental group generators are chosen as in Fig. 1. To prove (20) we notice that each loop γ_j , $1 \leq j \leq 3$, first crosses the

interval (t_{j-1}, t_j) from H to the lower half-plane, and crosses (t_j, t_{j+1}) back to H . The first crossing corresponds to the reflection σ_j and the second to the reflection in the circle $\sigma_j C_{j+1}$. This second reflection is $\sigma_j \sigma_{j+1} \sigma_j$, so the whole continuation around t_j is performed with the reflection

$$\sigma_j \sigma_{j+1} \sigma_j \sigma_j = \sigma_j \sigma_{j+1},$$

as stated.

If a projective monodromy representation satisfies (20) with some reflections σ_j , we say that this representation is *generated by reflections*. In Appendix I we will find the necessary and sufficient conditions for T_j to be generated by reflections, and will show how to find the σ_j when these conditions hold. We will see that the reflections σ_j are uniquely defined by the monodromy generators, except in the case when all these generators commute.

5. Special pentagons

The previous section motivates consideration of pentagons with one angle equal to 2, and the sides forming this angle mapped into the same circle by the developing map, while each pair of sides meeting at one of the other corners is mapped by the developing map to distinct circles.

We call them *special pentagons* and use the following notation

$$Q = (\overline{D}, t_1, t_2, t_3, t_4; q, f),$$

where t_j are naturally ordered corners with angles α_j , while q is the corner with angle 2 which can lie on any arc between the t_j , and the sides meeting at q are mapped by f to the same circle, while the circles containing $f([t_{j-1}, t_j])$ and $f([t_j, t_{j+1}])$ are distinct for all $j \in \mathbf{Z}_4$.

This notation is slightly inconsistent with our general notation for a circular pentagon, because only t_j are listed in their natural order, while q can be on any interval between them. To stress this, we separate q from the t_j by a semi-colon.

We recall that a *conformal quadrilateral*⁶ is a simply connected Riemann surface, which is conformally equivalent to a disk, with 4 marked prime ends⁷ Conformal equivalence of conformal quadrilaterals means the existence of a

⁶Not to be confused with circular quadrilateral!

⁷For a Jordan region in the plane prime ends are just boundary points. For general simply connected regions we refer to [1] and Appendix III.

conformal map between them which maps the marked points to the marked points.

Each conformal quadrilateral is conformally equivalent to a rectangle whose marked boundary points are the corners.

We consider special pentagons $Q = (\overline{D}, t_1, \dots, t_4; q, f)$ as conformal quadrilaterals $(\overline{D}, t_1, \dots, t_4)$, (forgetting the corner q) and define the *modulus*

$$\text{mod } Q \in (0, \infty)$$

as the extremal distance in D between the segments $[t_1, t_2]$ and $[t_3, t_4]$ of ∂D . For the definition and general properties of the extremal distance we refer to [1].

To avoid confusion with the sides of a pentagon as defined before, we use the word *segments* to denote $[t_j, t_{j+1}] \subset \partial D$. Thus one of the segments consists of two sides of the pentagon and contains q , while each of the other three segments is the closure of one side of the pentagon.

Every conformal quadrilateral is equivalent to $(\overline{H}, 0, 1, x, \infty)$ for some $x \in (1, \infty)$. With our convention that $(t_1, t_2, t_3, t_4) = (0, 1, x, \infty)$, the modulus is a strictly increasing function of x , mapping $(1, +\infty)$ onto $(0, +\infty)$. An explicit expression of this function can be found in [1] but we do not need this formula.

To state the properties of the extremal distance that we need, we use the intrinsic distance on \overline{D} defined in section 3.

Lemma 1. ([13, Lemma 13.1] and Lemma A4 in Appendix III.) *Consider a sequence of special pentagons Q_n whose areas are bounded from above. If the intrinsic distance between $[t_1, t_2]$ and $[t_3, t_4]$ tends to zero, while the intrinsic distance between $[t_2, t_3]$ and $[t_4, t_1]$ stays away from zero, then $\text{mod } Q_n \rightarrow 0$. If the intrinsic distance between $[t_2, t_3]$ and $[t_4, t_1]$ tends to 0 while the intrinsic distance between $[t_1, t_2]$ and $[t_3, t_4]$ stays away from zero then $\text{mod } Q_n \rightarrow \infty$.*

6. Evolution of special pentagons. Local families

We recall that $f(q)$ is called the *tip* of the slit. The slit can be lengthened or shortened with $f(q)$ moving on a circle. Lengthening or shortening the slit along the circle while keeping all circles of the chain unchanged, we obtain a one-parametric family of special pentagons, parametrized by some interval. We choose the length of the slit as parameter.

Lemma 2. *As a function of the length of the slit, $\text{mod } Q$ is monotone. It is strictly increasing if $q \in (t_2, t_3) \cup (t_4, t_1)$ and strictly decreasing if $q \in (t_1, t_2) \cup (t_3, t_4)$.*

This follows from the standard properties of the modulus, [1, 4.3] and Lemma A3 in Appendix III.

As the slit shortens, it eventually vanishes, and we obtain a polygon with at most 4 sides. As the slit becomes longer, it eventually hits the boundary and becomes a cross-cut which splits \overline{D} into two polygons.

Such a family, obtained from a special pentagon by shortening the slit until it vanishes and lengthening the slit until it hits the boundary, will be called a *local family* of special pentagons. It is parametrized by an open interval (for example, the length of the slit), and corresponds to an open interval on the ray $(1, +\infty)$ in view of Lemma 2.

In the remainder of this section we will study in detail what happens at the ends of a local family. In the next section we will see how local families are combined into a *global family* of special pentagons, parametrized by $x \in (1, +\infty)$, so that the special pentagons of the global family depend continuously and even real-analytically on x .

Consider a local family of special pentagons Q_x parametrized by $x \in I$ where I is an interval in $(1, \infty)$.

We say that the modulus *degenerates* if $\text{mod } Q_x \rightarrow 0$ or $\text{mod } Q_x \rightarrow \infty$ as x tends to an endpoint of I . This means that this endpoint must be 1 or ∞ .

First we state the conditions of degeneracy.

Suppose that $q \in (t_{k-1}, t_k)$. Suppose that the slit shortens and vanishes, then q must collide with a corner t_{k-1} or t_k . If the intrinsic length of $[t_{k-1}, q]$ is strictly smaller than the intrinsic length of $[q, t_k]$ then q will collide with t_{k-1} . If the intrinsic length of $[t_{k-1}, q]$ is strictly greater than the intrinsic length of $[q, t_k]$ then q will collide with t_k . In both cases we obtain a non-degenerated quadrilateral in the limit, so x tends to some $x_0 \in (1, \infty)$, and x_0 is a special point with the value $q(x_0) = t$, where $t \in \{t_{k-1}, t_k\}$ is the corner with which q collided. If the intrinsic lengths of $[t_{k-1}, q]$ and $[q, t_k]$ are equal, then as the slit shortens and vanishes, t_{k-1} and t_k collide, and the limit polygon is a triangle.

The degeneracy condition is thus the following:

D1. When the slit shortens and vanishes, $\text{mod } Q_x$ degenerates if and only if two corners collide in the limit.

In other words, our special pentagon must be a slit triangle with the slit originating at some vertex A . Notice that the angle of the triangle at A may be an integer, and the images of sides of the triangle which are adjacent at A may belong to the same circle.

Now suppose that the slit lengthens. Then eventually it will hit the boundary from inside at some point $s \in \partial H$.

*This means that $f(s)$ belongs to the circle of the slit
and the intrinsic distance between q and s tends to zero.* (21)

Suppose that $q \in (t_{k-1}, t_k)$. If $s \in [t_{k+1}, t_{k+2}]$, then the modulus degenerates, otherwise it does not. So we have the second degeneration condition:

D2. When the slit lengthens and splits the pentagon, $\text{mod } Q_x$ degenerates if and only if the slit hits the segment which is opposite to the segment to which q belongs.

In other words, in the limit, the slit splits the boundary into two arcs, and the modulus degenerates if and only if the closures of these two arcs contain at least two corners each.

Now we consider non-degenerate cases, that is the cases when there exists a limit quadrilateral when the slit vanishes or when it splits the pentagon.

Case 1. The slit vanishes, q collides with exactly one corner, and the angle of the special pentagon at this corner is positive.

Case 2. The slit lengthens and hits the boundary at an interior point of the side, splitting the special pentagon into a quadrilateral and a digon with positive angle.

It is also possible that as the slit lengthens, it hits the boundary at a corner. If the modulus does not degenerate, this must be a corner neighboring q , for example t_k . Then the special pentagon splits into a non-degenerate quadrilateral and the remaining part which can have only one corner at t_k . Therefore the detached part must be a disk. This we call

Case 3. The slit lengthens and hits the boundary at a corner. A disk splits away from the special pentagon, leaving a non-degenerate quadrilateral.

The remaining cases happen when the slit vanishes as in Case 1, and the corner with which the slit collides has zero angle, or when the slit lengthens, splits the special pentagon, and one part of the split pentagon is a digon with zero angle. These cases will be considered in the next section.

7. Transformations connecting local families into a global family.

In this section we explain what happens when x passes a special point x_0 , and q passes one of the t_j , so that Q_{x_0} is a non-degenerate quadrilateral.

We describe four types of transformations that may occur. The first three correspond to cases 1-3 of the previous section, and the 4-th transformation to the two remaining cases with a zero angle.

Transformation 1 (Fig. 2). Vanishing slit, Case 1.

Suppose that x passes a special point x_0 . Before this $q \in (t_{k-1}, t_k)$, and when $x = x_0$, $q(x_0) = s$, where s is one of the points t_{k-1} , t_k . After x passes x_0 , q and s interchange. As the images of the sides lie on the same fixed circles, we have the situation shown in Fig. 2.

The slit whose image was an arc of a circle vanishes, and then a new slit starts growing with the image on an arc of the circle that is adjacent to the previous circle at the corner where the old slit vanished.

The angle α of the limit quadrilateral at the corner where the slit vanishes satisfies

$$\alpha > 1. \quad (22)$$

Transformation 2 (Fig. 3). The slit hits an interior point s of a segment (see (21)). The slit is not tangent to this segment (Case 2).

If the modulus does not degenerate, s must be an interior point of the segment adjacent to that segment which contains q . So there is exactly one corner t in the interior of one boundary arc Z between q and s , and three corners on the complementary arc. When q hits s , our pentagon splits into two polygons: a quadrilateral with the corners s and $t_j \neq t$, and a digon with the corners s and t . Notice that the angles at the two corners of a digon are always equal. It is clear that in the described situation this digon angle is less than 1: it is the inclination of the slit to the side that it hits. Thus if α is the angle of the limit quadrilateral at s then

$$\alpha < 1, \quad (23)$$

while the digon has two angles $1 - \alpha$, at s and at t .

When the slit hits the boundary at an interior point of the side, a digon is detached, and a vertical digon⁸ is attached on the side which was hit. One side of the old slit becomes a side of the new pentagon.

Transformation 3 (Fig. 4). The slit hits a corner (see (21)) as described in Case 3.

In this case s is a corner, and there is no other corner on Z . When q hits s , the special pentagon splits into a quadrilateral with corners at t_j , $1 \leq j \leq 4$ and the other part which must be a disk. So if the limit quadrilateral has angle α at s then the special pentagon before the limit has angle $\alpha + 1$ at s .

So far we ignored the non-generic cases which may occur when some circles of the chain are tangent: when the slit vanishes at a corner with zero angle, and when the slit hits from inside a side which is tangent to it. In these cases one more transformation occurs.

Transformation 4 (Fig. 5). The slit vanishes at a corner with zero angle and a digon with zero angle is attached.

The slit shortens and vanishes at the corner with zero angle (which is shown at ∞ in Fig. 5a, and the resulting quadrilateral in Fig. 5b has angle 1. After that, we attach to this quadrilateral a digon with zero angle (shown as a strip in Fig. 5c, and the slit shortens when x continues to change in the same direction.

When we run x backwards, we first encounter Fig. 5c with the lengthening slit which hits the boundary of the pentagon from inside under zero angle. Similarly to transformation 2, a digon detaches (the strip in Fig. 5c is a digon with zero angle), and a new slit starts growing as in Fig. 5a.

Notice that unlike in all other transformations 1-3, the direction of the slit evolution (whether it lengthens or shortens) does not change for this transformation. Two other distinctions of this transformation from transformations 1-3 are that the old and new slit are on the same circle, and that q is on the same segment before and after the transformation.

This is consistent with the fact that the special points of the function $q(x)$ are simple, unless the angle of the special pentagon corresponding to a special point is zero, in which case this special point is a simple critical point for $q(x)$ [20, Ch. 9, §46]. The process we described shows that

⁸A pair of digons with equal angles formed by two circles are called *vertical*.

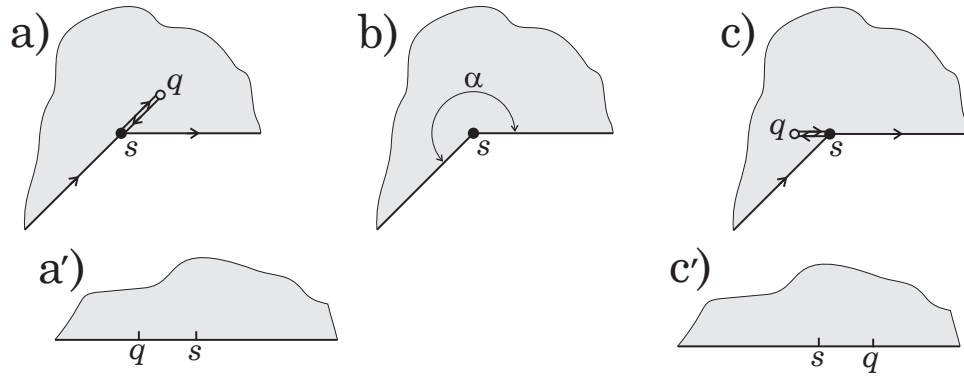


Figure 2: Transformation 1.

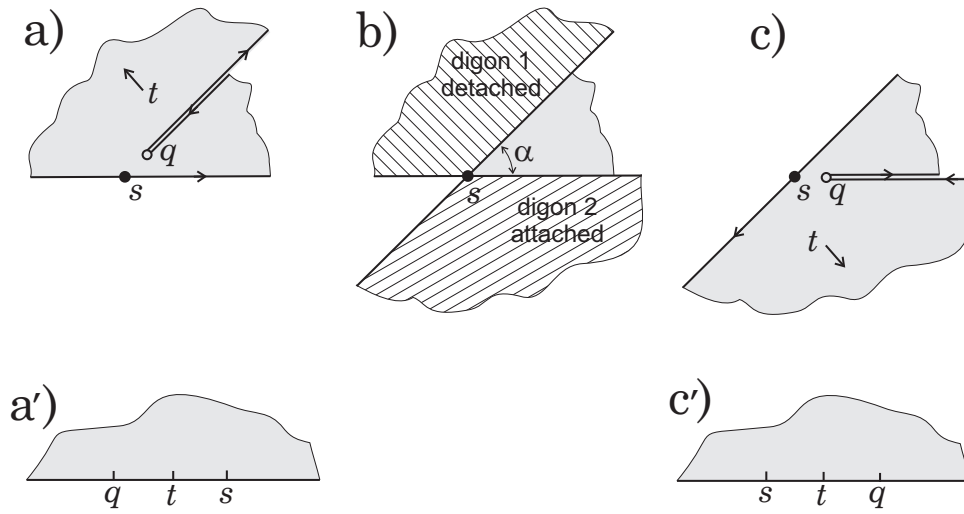


Figure 3: Transformation 2.

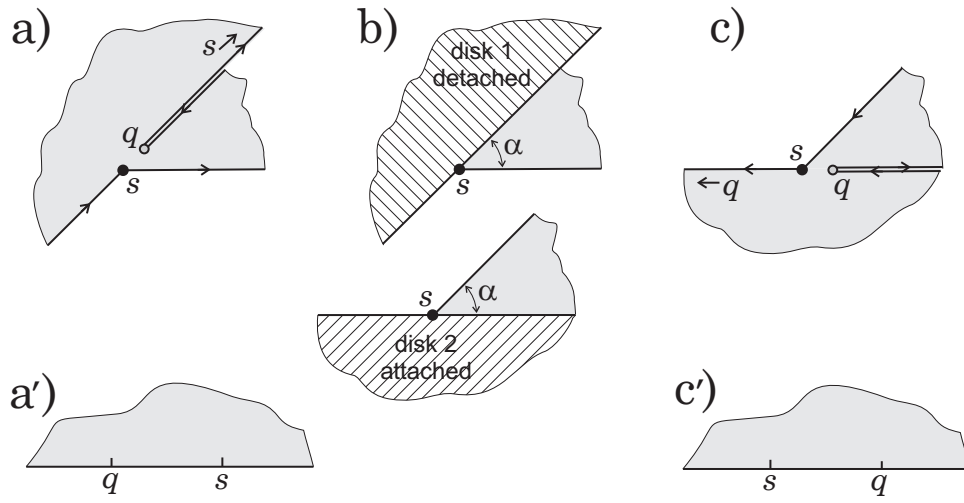


Figure 4: Transformation 3.

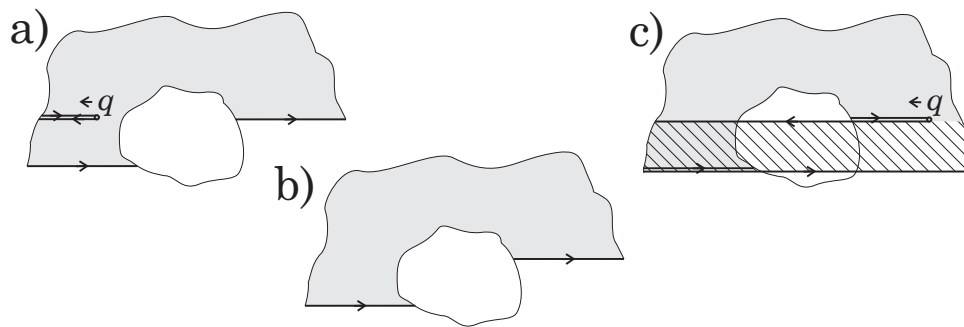


Figure 5: Transformation 4.

every local family of special pentagons can be extended to a global family of special pentagons, with the special pentagons becoming quadrilaterals at isolated points. At these points one angle α of the pentagon becomes angle β of the quadrilateral, and these angles are related as follows: $\beta = \alpha + 1$ for transformation 1, $\beta = \alpha - 1$ for transformation 3, and $\beta = 1 - \alpha$ for transformation 2.

This continuation can be either performed indefinitely in one or both directions, or the modulus can degenerate at one or both ends.

In sections 8–10 we will analyse global families.

Remark. Transformations 1–4 suggest the following:

When equation (4) undergoes an isomonodromic deformation and q collides with some t_k , then the resulting Heun equation has exponent difference $\pm(\kappa_k + 1)$ at t_k as in Case 1, or $\pm(\kappa_k - 1)$ as in Cases 2,3.

This is true in general, without our restriction that the κ_j and x are real. To obtain this result one can use asymptotics of $p(x)$ and $q(x)$ as x tends to a special point written in [36, p. 534–535], and obtain the limit equation with four singularities directly from (4).

8. Explicit description of global families

The previous section shows how local families are combined into a global family. A global family $Q(x)$, $1 < x < +\infty$ consists of local families of special pentagons parametrized by intervals $x_j < x < x_{j+1}$. At the points $x_j \in (1, \infty)$ the special pentagon becomes a non-degenerate quadrilateral. A global family may consist of a single local family; such families will be discussed in section 10. The sequence x_j can be finite or infinite in one direction or in both directions. The smallest and the largest terms of this sequence, when they exist, are 1 and ∞ . All other terms correspond to the special points of the solution of PVI which is described by our global family. To describe global families more precisely, we recall the construction of combinatorial objects related to circular polygons.

9. Representation of polygons by nets.

Circular polygons are conveniently represented by nets [9, 13, 14]. Consider a polygon given by (15). Its *net* is the cell decomposition of \overline{D} by all f -preimages of the circles C_k , where C_k is the circle that contains $f([t_{k-1}, t_k])$. The corners are required to be vertices of the net. The 1-cells of the net are

labeled by their images. Two nets are considered equal if there is an orientation preserving homeomorphism which maps one onto another preserving the labels of the corners. Let e_1 be the 1-cell on the boundary of the net, oriented according to the orientation of the boundary, and beginning at t_1 . Two polygons with developing maps f_1, f_2 are equal if their circles C_j are the same, their nets are equal, and the images e_1 , as oriented 1-cells, are equal.

In the illustrations we label the circles C_j and corresponding 1-cells of the nets with colors (or with different styles of lines in the black and white version).

It is difficult to characterize all possible nets of circular quadrilaterals or special pentagons. The topological classification of generic 4-circle chains is given in Appendix II. For each type of chain, one has a set of nets compatible with this chain. For the chain topologically equivalent to a quadruple of generic great circles as in Fig. 8, one can give the following characterization of the nets. Notice that the cell decomposition of the sphere in Fig. 8 has the following property which must be inherited by the net:

a) When two 2-cells share a boundary 1-cell, one of these two cells is a quadrilateral and another is a triangle.

Moreover, the net has an evident additional property:

b) All interior vertices have degree 4, and all vertices on the sides have degree 3.

Our standing assumption (11) implies that

c) The degrees of the corners (as vertices of the net) are even.

These three properties completely characterize the nets of circular quadrilaterals over the 4-circle chain shown in Fig. 8. See also Fig. 23A, where the same 4-circle chain is shown.

Thus for example, all cell decompositions in the right column of Fig. 20 are nets of quadrilaterals.

10. Real solutions of PVI without special points

Our paper [11] describes all complex solutions of PVI without special points in the complex plane.

In this section we will describe all real solutions of PVI with real parameters, which have no real special points.

For simplicity we limit ourselves to the generic case: all parameters κ_j are

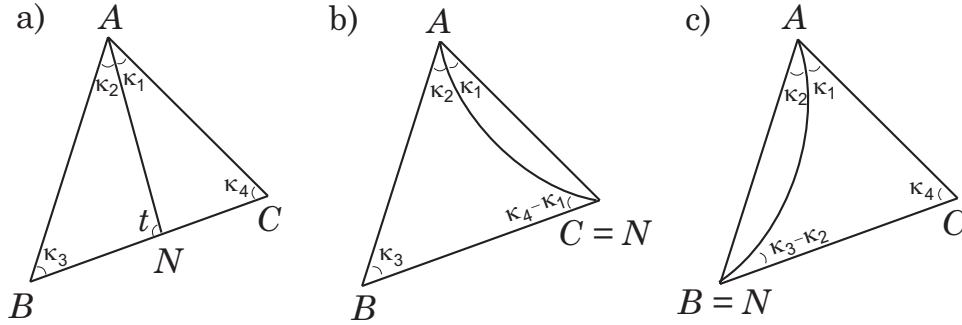


Figure 6: Slit triangles.

not integers, and the circles of the chain are not tangent to each other. The last condition holds for example when the projective monodromy contains no parabolic transformations.

Solutions without special points correspond to local families for which the modulus degenerates on both ends. So degeneracy conditions D1 and D2 of section 6 must be satisfied (one condition on one end and another on another end).

Thus we have one of the three configurations shown in Fig. 6.

In a family without special points we must have

$$q(x) \in (t_j, t_{j+1}) \quad \text{for all } x \in (1, \infty)$$

and some $j \in \mathbf{Z}_4$. Suppose without loss of generality that $j = 1$, so that

$$q(x) \in (0, 1), \quad x > 1. \quad (24)$$

The condition that $f(0) = f(1)$ when the slit vanishes can be stated in terms of projective monodromy representation:

$$\begin{aligned} &\text{Transformations } T_0 \text{ and } T_1 \text{ have a common fixed point} \\ &\text{with multipliers } e^{2\pi i \kappa_1} \text{ and } e^{2\pi i \kappa_2}. \end{aligned} \quad (25)$$

We will use the results of Klein [29] and Van Vleck [41] on circular triangles. First of all we have

Lemma 3. *For any positive numbers λ_j , $1 \leq j \leq 3$, there exists a unique equivalence class of circular triangles with angles $(\lambda_1, \lambda_2, \lambda_3)$.*

Sketch of the proof. The developing map of a triangle with angles λ_j satisfies the Schwarz differential equation:

$$\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 = \frac{1 - \lambda_1^2}{2z^2} + \frac{1 - \lambda_2^2}{2(z-1)^2} + \frac{\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - 1}{2z(z-1)},$$

see, for example, [24, p. 452] or [18, Ch. VI, §3]. Parameters λ_j can be arbitrary non-negative numbers, and for fixed parameters all solutions give equivalent triangles. This proves the lemma.

Lemma 4. *In a triangle with angles $(\lambda_1, \lambda_2, \lambda_3)$, the image of the closure of the side opposite to λ_1 under the developing map makes*

$$E \left(\frac{\lambda_1 - \lambda_2 - \lambda_3 + 1}{2} \right)$$

full turns⁹ around the circle containing this image. Here $E(x)$ is the integer part of x for $x > 0$ and zero otherwise.

This is called the Ergänzungsrelation of Klein [29], [41].

First we address the easier cases b) and c) in Fig. 6. Consider the case b). In this case, our triangle ABC is split into two parts, one of which is a digon with both angles κ_1 , and the other part is a triangle with angles $(\kappa_2, \kappa_3, \kappa_4 - \kappa_1)$. The image of a side of a digon cannot cover a full circle. Therefore the image of the side NA of the triangle cannot cover the full circle. The necessary and sufficient condition for this according to Lemma 4 is

$$\kappa_3 + \kappa_1 \leq \kappa_2 + \kappa_4 + 1. \quad (26)$$

Similarly, Fig. 6c produces the condition

$$\kappa_4 + \kappa_2 \leq \kappa_1 + \kappa_3 + 1. \quad (27)$$

As there are no free parameters in the configurations in Fig. 6b,c, we conclude that when (26) is satisfied, there is a single equivalence class of configurations of the form Fig. 6b and when (27) is satisfied, there is a single equivalence class of configurations of the form Fig. 6c with these angles. This means that each PVI with such parameters has an isolated solution of type b) or c), or

⁹When $f : [a, b] \rightarrow C$ is an immersion of a closed interval $[a, b]$ into a circle C then the “number of full turns” is defined as $\text{card } f^{-1}(a) - 1$.

both. Notice that the two inequalities (26) and (27) cover the whole range of real parameters, so we conclude that isolated solutions of PVI of one or both types Fig. 6b,c always exist. There can be one or two of them.

Now we turn to the case a). We introduce the auxiliary angle $t \in [0, 1]$ as a parameter (see Fig. 6a).

When t is fixed, there exist two equivalence classes of triangles with prescribed angles: NAB with angles (t, κ_2, κ_3) and ANC with angles $(\kappa_1, 1 - t, \kappa_4)$. Let f_1 and f_2 be their developing maps.

The question is when we can glue such two triangles along the side NA . As we assume that the circles of the chain are not tangent to each other, $t \in (0, 1)$, and we can post-compose the f_j with linear-fractional transformations to achieve $f_j(N) = 0$, $f_1(BN)$ and $f_2(NC)$ belong to a line ℓ through the origin, and $f_j(NA)$ is contained in the real line for $j = 1, 2$. Then it is clear that the necessary and sufficient condition for the possibility of gluing is that the image of the closed side NA under both developing maps intersects the line ℓ the same number of times. Indeed, if this is so, the images of the point A under both developing maps are on the same side of the line ℓ (or both at 0, or both at ∞), and these images can be made equal by additional scaling $z \mapsto rz$ with some $r > 0$.

As ℓ intersects the real line at 0 and ∞ we are interested in the combined numbers of zeros and poles of f_j on NA .

Consider triangle NAB and denote by $(\lambda_1, \lambda_2, \lambda_3)$ the angles at (N, A, B) . To count the number of zeros and poles of f_1 on NA times the image of AN intersects the circle that contains NB we use the results of Klein [29] (see also [41]) on the number of zeros of hypergeometric function on an interval $[0, 1]$.

We recall that the hypergeometric function $w(z) = F(\alpha, \beta, \gamma, z)$ is the solution of the hypergeometric equation

$$z(z-1)\frac{d^2w}{dz^2} - (\gamma - (\alpha + \beta + 1)z)\frac{dw}{dz} + \alpha\beta w = 0, \quad (28)$$

which satisfies $F(0) = 1$ and is holomorphic at 0. A second linearly independent solution of the same equation is

$$F_1(z) = z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z).$$

Thus F/F_1 is a developing map of a triangle whose angles are the absolute values of the exponent differences of (28),

$$\lambda_1 = |1 - \gamma|, \quad \lambda_2 = |\gamma - \alpha - \beta|, \quad \lambda_3 = |\alpha - \beta|.$$

We choose

$$\lambda_1 = 1 - \gamma, \quad \lambda_2 = \alpha + \beta - \gamma, \quad \lambda_3 = \alpha - \beta,$$

which defines α, β, γ uniquely. Notice that $F_1(0) = 0$ because $\gamma \in (0, 1)$. Then $f_1 = F/F_1$ is the developing map of NAB , and the side $NA = (0, 1)$.

The number of crossings between $f_1(AN)$ and ℓ is equal to the combined number of zeros and poles of F and F_1 on $[0, 1]$. We only consider the case $\lambda_1 \in (0, 1)$ which we need.

According to [41] the number of zeros of F on $[0, 1]$ is:

- (i) zero, if $[\lambda_2] > [\lambda_3]$,
- (ii) $E((\lambda_1 + \lambda_3 - \lambda_2 + 1)/2)$, if $[\lambda_3] > [\lambda_2]$,
- (iii) 0 or 1 depending on whether $E((\lambda_1 + \lambda_3 - \lambda_2 + 1)/2)$ is even or odd, if $[\lambda_2] = [\lambda_3]$.

The number of zeros of F_1 is always

$$E\left(\frac{\lambda_3 - \lambda_1 - \lambda_2 + 1}{2}\right) + 1.$$

Adding these together we obtain that the number of crossings between the image of AN and the circle of BN equals to:

1, if $[\lambda_2] \geq [\lambda_3]$,

and to

$$E\left(\frac{\lambda_1 + \lambda_3 - \lambda_2 + 1}{2}\right) + E\left(\frac{\lambda_3 - \lambda_1 - \lambda_2 + 1}{2}\right) + 1, \quad \text{if } [\lambda_2] < [\lambda_3].$$

Applying this result to NAB and the similar result to CAN , (or rather to its mirror image), we obtain that the gluing is possible if and only if one of the following two conditions holds:

- 1) $\kappa_2 \geq \kappa_3$ and $\kappa_1 \geq \kappa_4$,
- 2) $\kappa_2 < \kappa_3$ or $\kappa_1 < \kappa_4$, and

$$E\left(\frac{\kappa_3 - \kappa_2 + t + 1}{2}\right) = E\left(\frac{\kappa_4 - \kappa_1 - t + 2}{2}\right), \quad (29)$$

and

$$E\left(\frac{\kappa_3 - \kappa_2 - t + 1}{2}\right) = E\left(\frac{\kappa_4 - \kappa_1 + t}{2}\right), \quad (30)$$

To simplify these conditions, we put

$$t = u + 1/2, \quad -1/2 \leq u \leq 1/2,$$

$$a = \kappa_3 - \kappa_2 + 1/2, \quad b = \kappa_4 - \kappa_1 + 1/2.$$

Then conditions 1)-2) become

1') $a \leq 1/2$, $b \leq 1/2$, or

2') $E((a + 1 + u)/2) = E((b + 1 - u)/2)$, and $E((a - u)/2) = E((b + u)/2)$.

Eliminating $u \in (-1/2, 1/2)$ we obtain:

$$|a - b| < 1, \quad \text{or} \quad \min\{a, b\} < 3/2, \quad (31)$$

the shaded region in Fig. 7. For these values of parameters, PVI has an interval of solutions without special points, in addition to one or two isolated solutions of types b), c). The boundaries of the regions corresponding to (26) and (27) are shown as lines $a = b + 1$ and $b = a + 1$ in Fig. 7.

Theorem 1. *A real solution $q(x)$ of PVI with real parameters κ_j defined on $(1, \infty)$ and satisfying $q(x) \in (0, 1)$, $x > 1$ always exists. The monodromy corresponding to this solution satisfies (25). If (31) with a and b as in (6), (7) holds, then there is an interval of such solutions. If (26) or (27) hold, then we have additional one or two isolated solutions.*

The cases $q(x) \in (t_j, t_{j+1})$ for $j = 2, 3, 4$ are obtained by a cyclic permutation of the angles in our conditions.

In [6], the following theorem is proved: *Solutions of PVI with parameters $(1/2, 1/2, 1/2, 1/2)$ corresponding to unitary monodromy do not have special points on a real interval between the fixed singularities.*

These authors do not assume a priori that their solutions are real. For the case of real solutions, this result can be obtained as follows.

Consider some special pentagon corresponding to a real solution of this equation. The angles are $(1/2, 1/2, 1/2, 1/2, 2)$, and the circles C_j are great circles. Removing the slit we would obtain a geodesic quadrilateral with

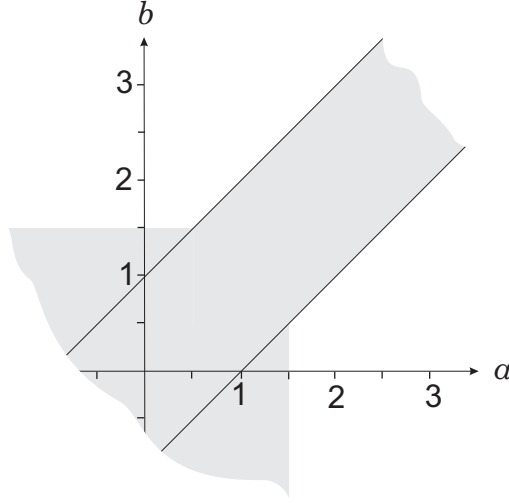


Figure 7: Conditions on a and b .

angles $(1/2, 1/2, 1/2, 3/2)$, or a triangle with angles $(1/2, 1/2, 1)$, but it is easy to see that such quadrilateral does not exist (see for example, [10]). So it must be a geodesic triangle with angles $(1/2, 1/2, 1)$. Then the special pentagon must have the shape as in Fig. 6a, so our global family does not have special points.

In [5], the following fact is proved: *Solutions of PVI with parameters $(1/2, 1/2, 1/2, 3/2)$ corresponding to unitary monodromy do not have poles on $\mathbf{R} \setminus \{0, 1\}$.* Again, in the case of real solutions, this follows from our results. When the special pentagon corresponding to such a solution undergoes a transformation with $q = \infty$ the limit quadrilateral must have angles $(1/2, 1/2, 1/2, 1/2)$ or $(1/2, 1/2, 1/2, 5/2)$. Geodesic quadrilaterals with such angles do not exist [10].

11. Some other special cases

We mention several special cases without going into detail.

1. Suppose that the projective monodromy representation is reducible. This means that all linear-fractional transformations T_j have a common fixed point. Without loss of generality, we place it at infinity. Then all monodromy transformations are affine, and the circles C_j of the chain are straight lines. Our pentagons are rectilinear and the developing maps can be expressed by

the Schwarz–Christoffel formula. Then $q(x)$ can be expressed in terms of hypergeometric integrals.

Indeed, the Schwarz–Christoffel formula of a rectilinear special pentagon gives

$$f(z) = c \int_0^z \zeta^{\alpha_1-1} (\zeta - 1)^{\alpha_2-1} (\zeta - x)^{\alpha_3-1} (\zeta - q) d\zeta = c(I_1(z) - qI_2(z)),$$

where

$$I_1(z) = \int_0^z \zeta^{\alpha_1} (\zeta - 1)^{\alpha_2-1} (\zeta - x)^{\alpha_3-1} d\zeta$$

and

$$I_2(z) = \int_0^z \zeta^{\alpha_1-1} (\zeta - 1)^{\alpha_2-1} (\zeta - x)^{\alpha_3-1} d\zeta.$$

Normalization $f(1) = a$ and $f(\infty) = b$ will define our special pentagon completely, so we obtain with $k = b/a$

$$q(x) = \frac{kI_1(1) - I_1(\infty)}{kI_2(1) - I_2(\infty)},$$

an expression for $q(x)$ in the form of hypergeometric integrals.

2. Suppose that one of the monodromy transformations is the identity, and the corresponding angle is 1. Then our special pentagon is in fact a slit triangle (or a slit digon, or a slit disk). In this case, the developing map itself can be expressed in terms of hypergeometric functions, see [40], where the case of slit-triangle quadrilaterals has been studied in great detail.

These are the two known cases of reduction of PVI when some solutions can be explicitly found. In a certain sense there are no other cases [43], except some cases [33] when a solution can be expressed as a somewhat non-standard combination of classical special functions.

12. Examples

We begin with the simplest examples when all special pentagons in a global family are regions in the sphere, so the nets are not required for their description.

Example 1. Consider the conformal map f from the upper half-plane onto the shaded region in Fig. 9a. This is the developing map of a special pentagon. The corners on ∂H are shown below, and their images are shown in parentheses.

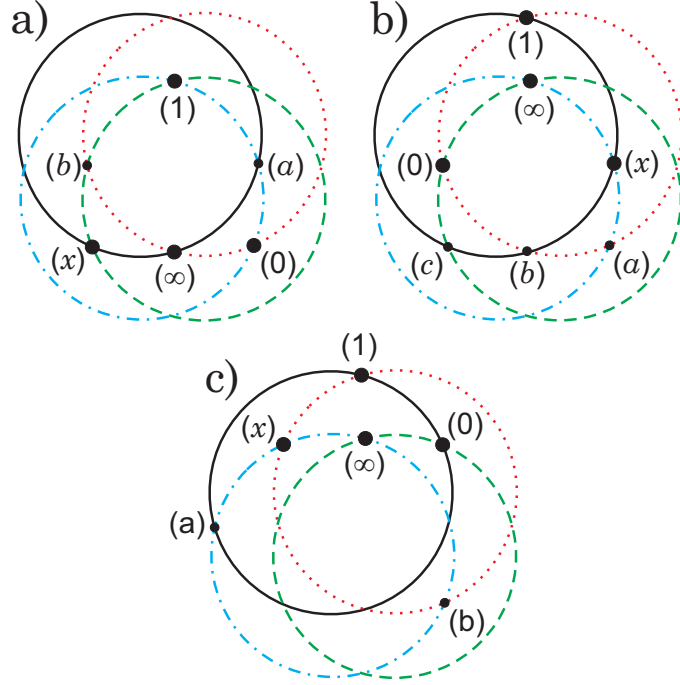


Figure 8: Four-circle chains for a) Example 1, b) Example 2, c) Example 7.

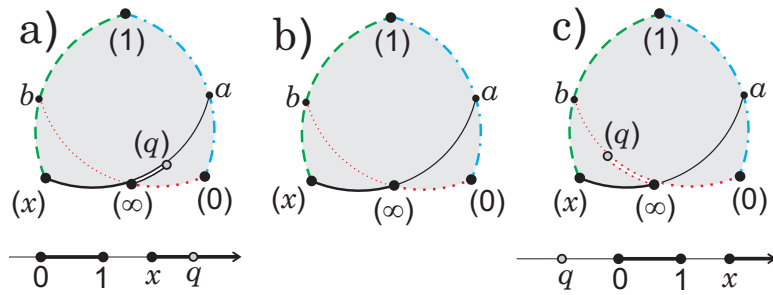


Figure 9: Global family in Example 1.

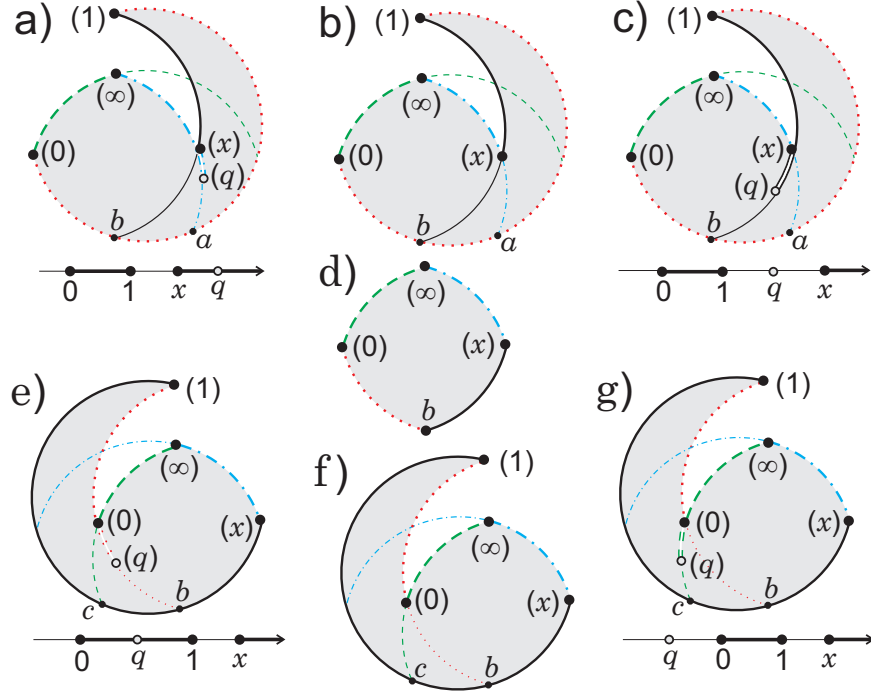


Figure 10: Global family in Example 2.

Suppose that the slit lengthens. Then the extremal distance between the segment $[(0), (1)]$ and the opposite segment $[(x), (\infty)]$ (which contains (q)) decreases, so the extremal distance in H between $[0, 1]$ and $[x, \infty]$ decreases, thus x decreases.

When (q) tends to $[(0), (1)]$, modulus degenerates and $x \rightarrow 1$.

When the slit shortens, x increases, and eventually the slit vanishes and we obtain Fig. 9b. At this moment $q = \infty$. We have transformation 1, so after that a new slit starts as shown in Fig. 9c and when it hits $[(1), (x)]$, the extremal distance between $[0, 1]$ and $[x, \infty]$ tends to infinity which implies that $x \rightarrow +\infty$.

We conclude that solution $q(x)$ of PVI which corresponds to this global family has one pole on $(1, \infty)$, and has no zeros, no fixed points and no 1-points.

The chain of circles corresponding to this example is shown in Fig. 8a. For example, this can be any four generic great circles, which corresponds to $SU(2)$ monodromy, determined by formula (20), see Appendix III.

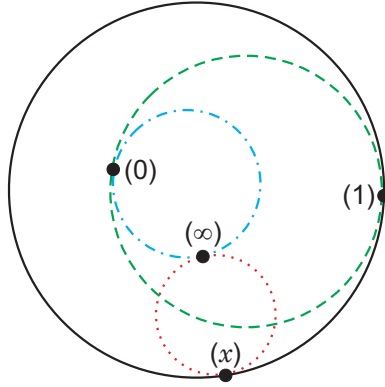


Figure 11: Circles chain for Example 3.

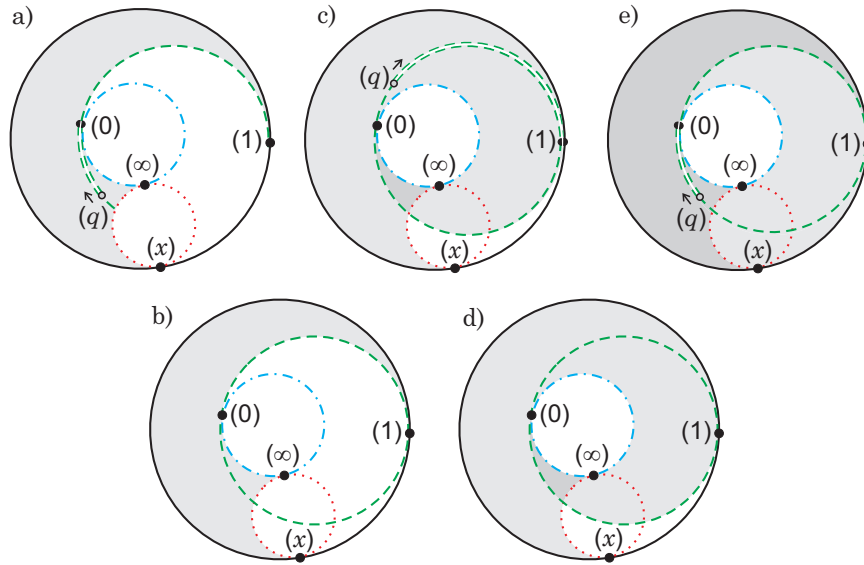


Figure 12: Example 3.

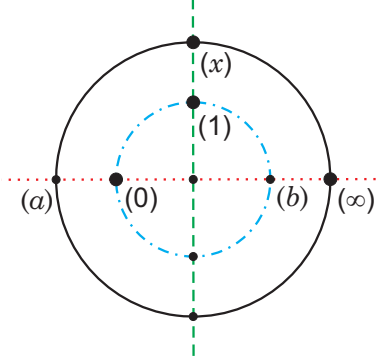


Figure 13: Four-circle chain for Example 4.

Example 2. In this example, Fig. 10, the image of the developing map is also a region in the sphere. In Fig. 10a, when the slit increases, the extremal distance between $[0, 1]$ and $[x, \infty]$ decreases which means that x decreases. When the slit hits the segment $[(0), (1)]$, this distance tends to zero, which means that $x \rightarrow 1$.

As the slit in Fig. 10a decreases, x increases, and when the slit vanishes we obtain Fig. 10b. At this moment $q(x) = x$, and we have transformation 1. As x increases further we have Fig. 10c, and then, when (q) hits $[(0), (1)]$, we obtain Fig. 10d. This is transformation 2, and $q(x) = 1$ at this point. The digon on the right of Fig. 10c was detached. According to the transformation 2, we attach to the quadrilateral in Fig. 10d the vertical digon shown on the left of Fig. 10e. The slit in Fig. 10e shortens as x increases. When it vanishes we obtain Fig. 10f where $q(x) = 0$, and we have transformation 2. After the transformation we obtain Fig. 10g, where the slit lengthens as x increases. Eventually the slit hits $[(1), (x)]$ which corresponds to $x \rightarrow +\infty$.

Therefore, the solution $q(x)$ in this example has three special points $x_1 < x_2 < x_3$ on $(1, +\infty)$, such that $q(x_1) = x_1$, $q(x_2) = 1$, $q(x_3) = 0$. These three special points correspond to quadrilaterals in Fig. 10b,d,f. Monodromy is determined by the four circles in Fig. 8b by formula (20).

Example 3. Consider the 4-circle chain shown in Fig. 11, where all pairs C_j, C_{j+1} are tangent. A quadrilateral, which is a subset of the sphere, is the shaded region in Fig. 12b. To obtain a special pentagon we make a slit $[(0), (q)]$ shown in Fig. 12a. When this slit lengthens, it eventually hits the segment $[(x), (\infty)]$ and modulus degenerates, $x \rightarrow 1$. As the slit shortens

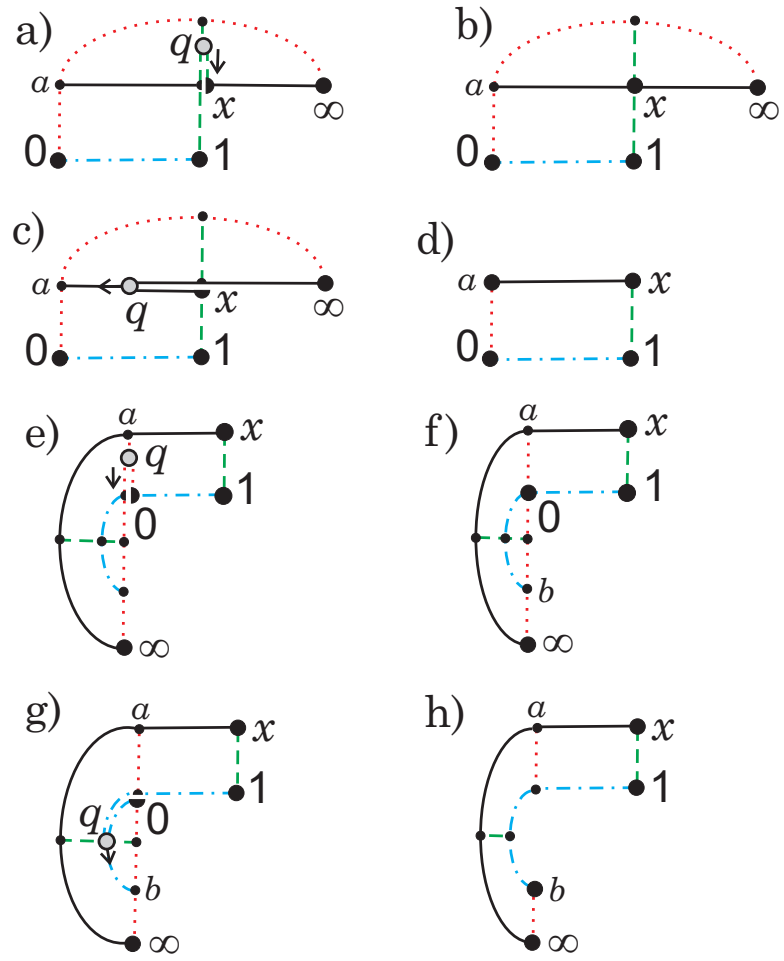


Figure 14: Nets for the global family in Example 4.

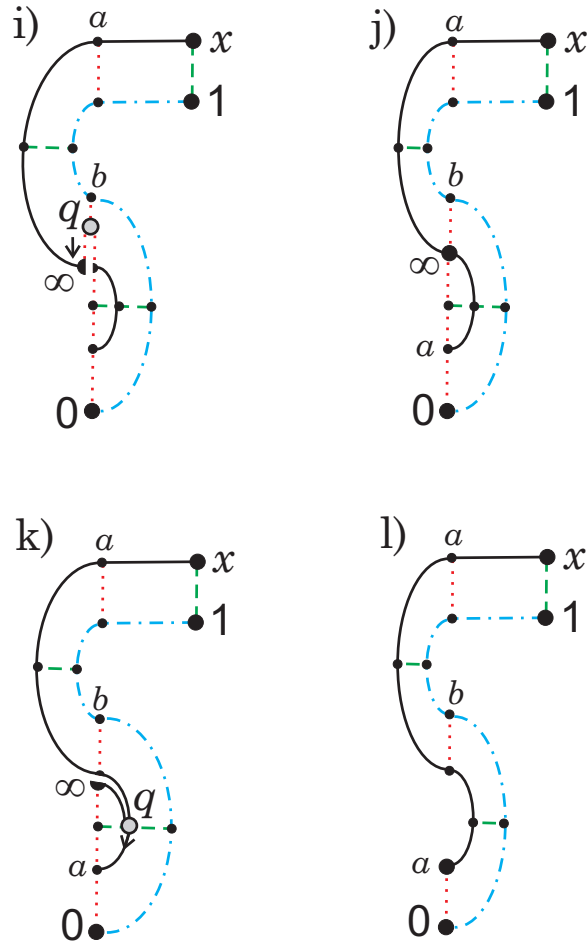


Figure 15: Nets for the global family in Example 4 (continued).

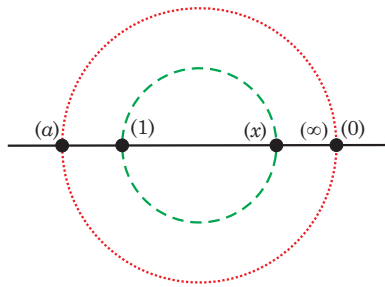


Figure 16: Three-circle chain for Example 5.

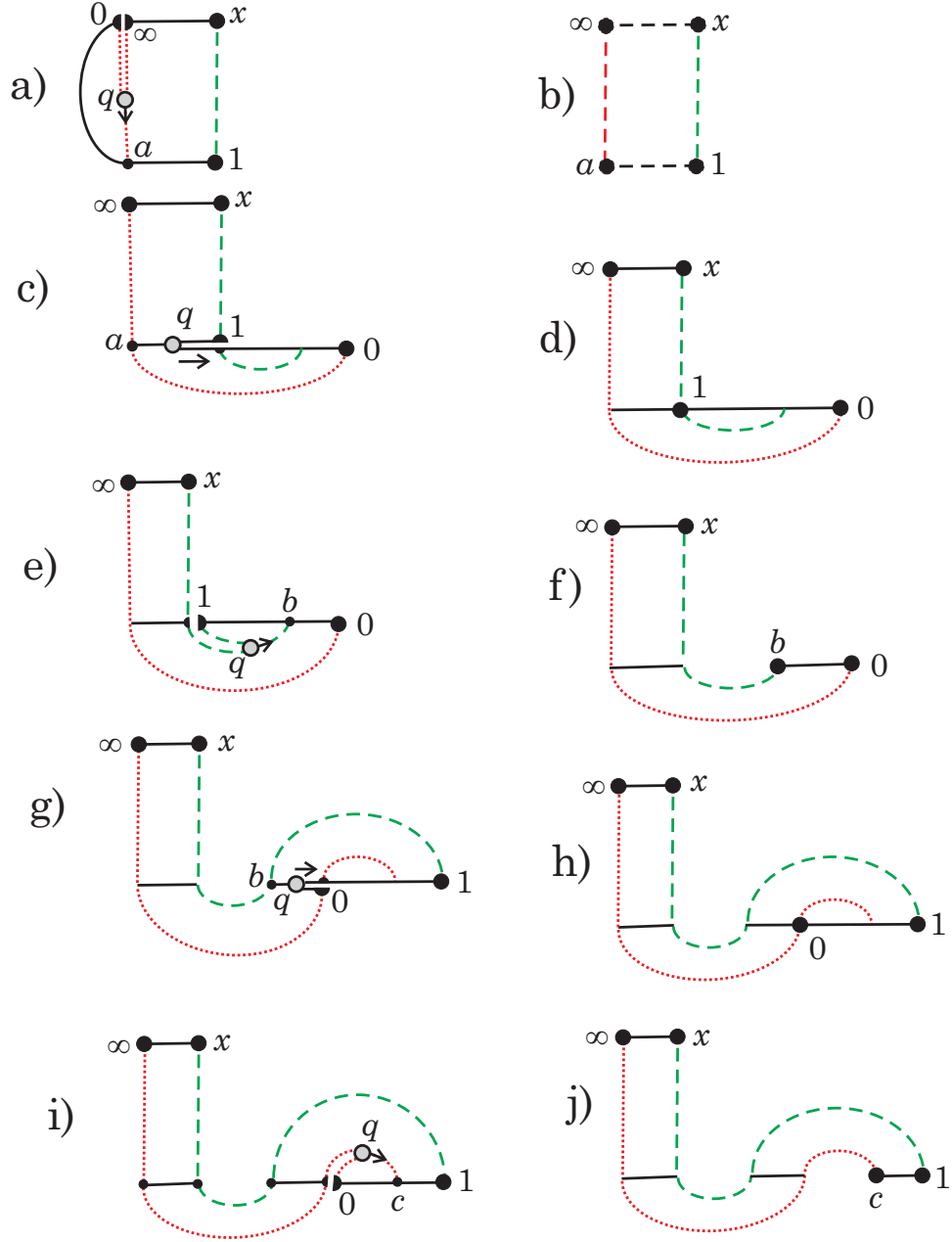


Figure 17: Nets for the global family in Example 5.

and vanishes in Fig. 12b, we have transformation 4. After that, a digon with zero angle is attached to the shaded region in Fig. 12b along a small arc $[(0), (q)]$ in Fig. 12c, and the new slit $[(q), (1)]$ continues to shorten. The special pentagon in this figure is not a subset of the sphere anymore: the dark shaded area is covered twice. When (q) hits (1) , the slit in Fig. 12c vanishes, and a new transformation 4 occurs at a quadrilateral shown in Fig. 12d. Afterwards, a sequence of transformations 4 continues indefinitely, alternately at (0) and (1) with the segments $[1, x]$ and $[\infty, 0]$ of the pentagon increasing by a full circle length after each two transformations, thus $q(x)$ oscillates between 0 and 1 as $x \rightarrow +\infty$. The sequence of special points is $(0, 1, (0, 1), \dots)$.

Example 4. Each special pentagon in this family is mapped by the developing map to a four-circle chain shown in Fig. 13. It is easy to check that Fig. 14b is a net corresponding to this chain. We make a cut as shown in Fig. 14a, and start a global family from this local family.

The nets for the global family are shown in Figs. 14 and 15, with the left columns (a,c,e,g,i,k) containing the local families of special pentagons, and the right columns (b,d,f,h,j,l) containing quadrilaterals connecting the local families. Modulus degenerates on one end ($x \rightarrow +\infty$). As $x \rightarrow 1$, we have an infinite chain of local families so as x increases we have the following sequence of special points:

$$(\dots, (\infty, \infty, 0, 0), \infty, x)$$

and no 1-points. The sequence has period of length 4, $(\infty, \infty, 0, 0)$ repeated infinitely many times on the left. The monodromy in this example is

$$M_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad M_t = \begin{pmatrix} 0 & Ri \\ Ri & 0 \end{pmatrix},$$

where we assume that the inner and outer circles have radii 1 and R .

Example 5. Parameters are the same as in the previous example but monodromy is different. The global family is shown in Fig. 17, with the left column (a,c,e,g,i,...) containing the local families of special pentagons, and the right column (b,d,f,h,j,...) containing quadrilaterals connecting the local families. The corresponding three-circle chain is shown in Fig. 16b. There is an infinite sequence of local families; as x increases from 1 to ∞ we have the following sequence of special points:

$$(0, (1, 1, 0, 0), \dots).$$

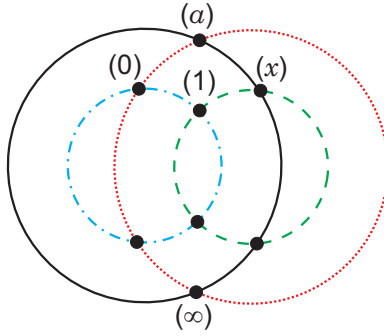


Figure 18: Four-circle chain for Example 6.

The sequence has period $(1, 1, 0, 0)$ repeated infinitely many times on the right.

Example 6. The circle chain (see Fig. 18) consists of two pairs of non-intersecting circles. The global family is a doubly infinite sequence. A part of this global family is shown in Fig. 19. It starts with a local family represented by a pentagon in Fig. 19a. As q tends to a , this pentagon degenerates to the quadrilateral shown in Fig. 19b. In the opposite direction, when q tends to x , the pentagon in Fig. 19a does not degenerate, and the sequence continues indefinitely, with the length of the sides $[1, x]$ and $[\infty, 0]$ ever increasing. At the other end of the sequence shown in Fig. 19, a quadrilateral in Fig. 19j is symmetric with respect to reflection preserving the vertices 1 and a'' and exchanging 0 with x . The sequence then continues by a local family reflection symmetric to the pentagon shown in Fig. 19i (with the direction of q reversed), and continues indefinitely, with the length of the sides $[0, 1]$ and $[x, \infty]$ ever increasing. We have a doubly infinite sequence of special points

$$(\dots (\infty, \infty, x, x), \infty, (0, 0, \infty, \infty), \dots).$$

The sequence is infinite in both directions, has period (∞, ∞, x, x) on the left and $(0, 0, \infty, \infty)$ on the right.

Example 7. Consider the 4-circle chain in Fig. 8c. To construct a global family, we begin with a quadrilateral represented by a net in the right column of Fig. 20. That all nets in this column represent some quadrilaterals follows from the criterion given in section 9.

Let us begin with the quadrilateral in Fig. 20b. To transform it to a special pentagon, we make a slit as in Fig. 20a. Lengthening of this slit

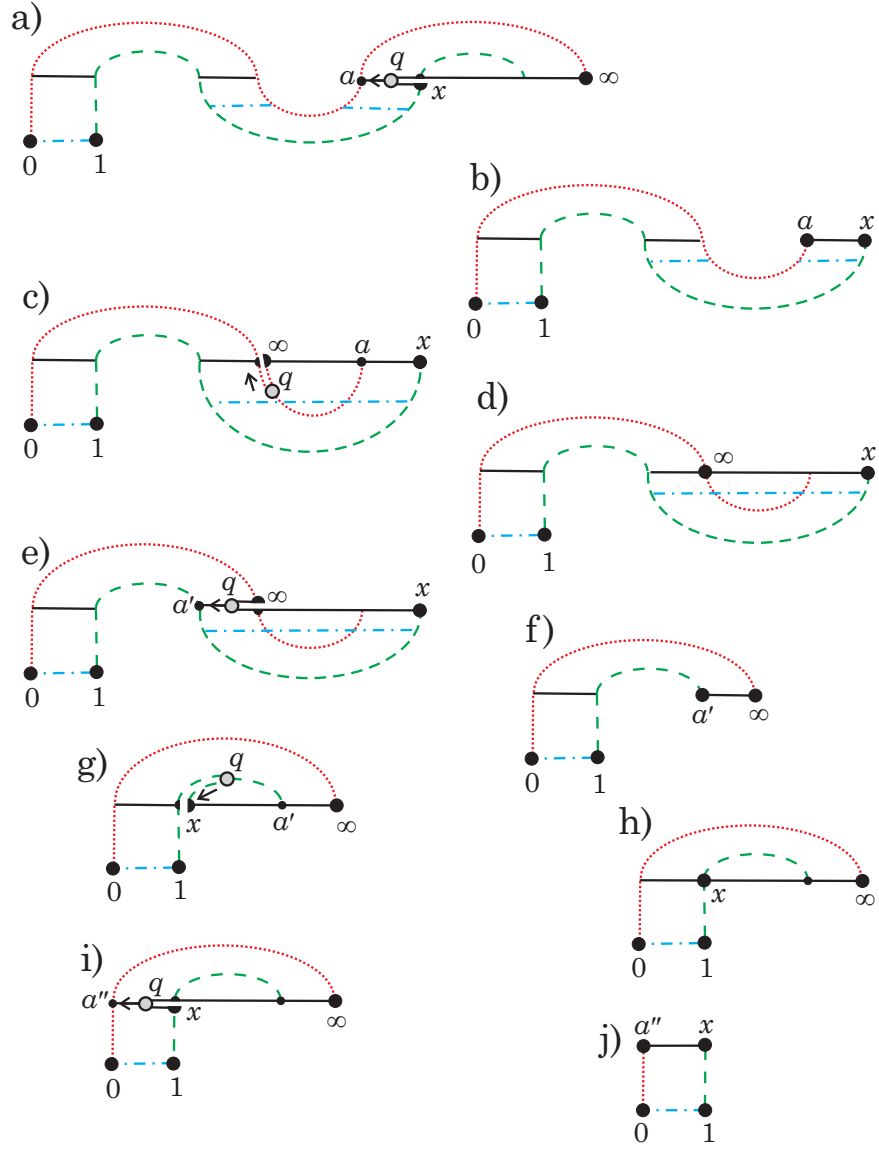


Figure 19: Nets for a part of the global family in Example 6.

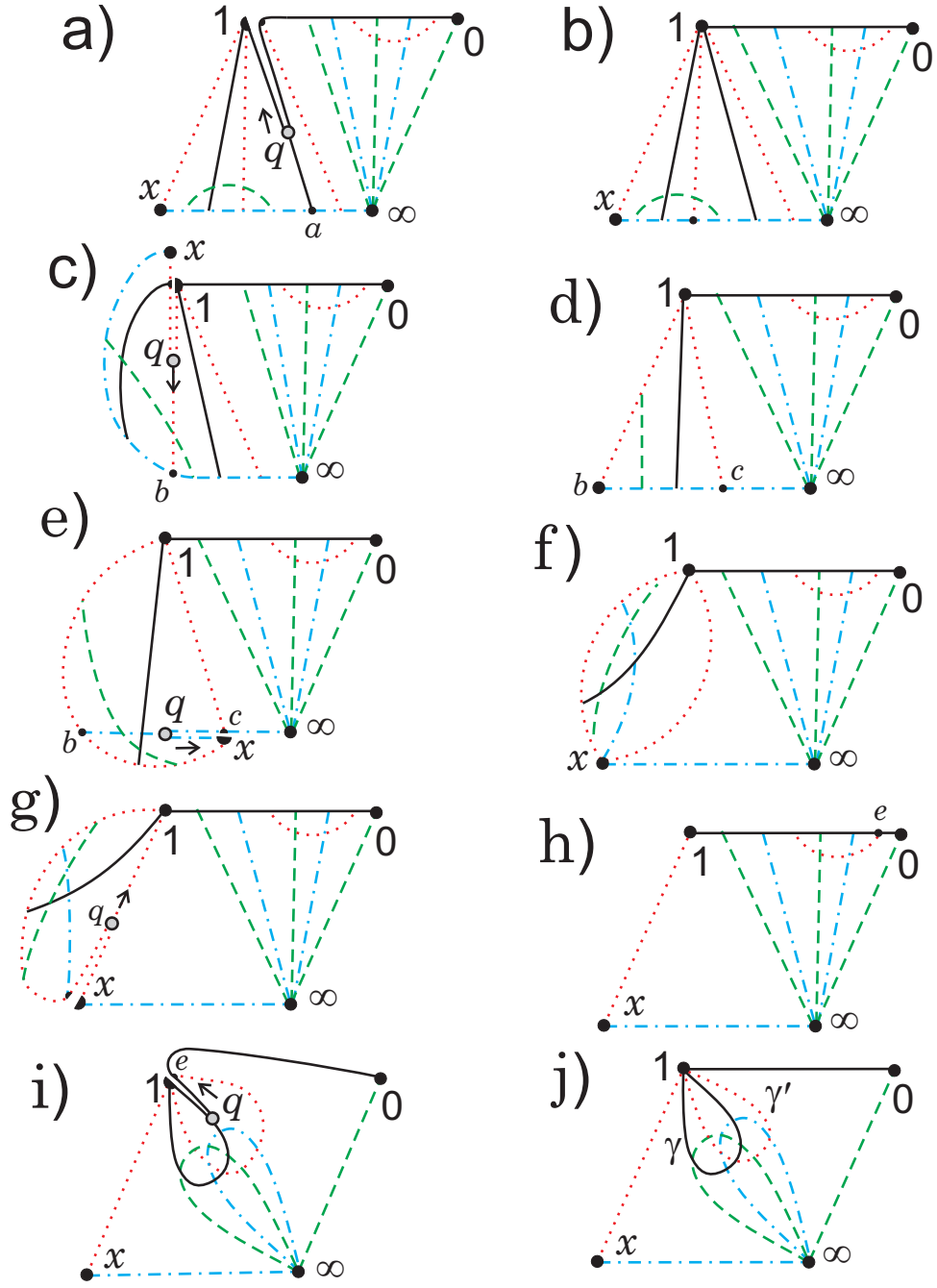


Figure 20: Nets for a half of the global family in Example 7.

corresponds to decreasing x . In particular, when the slit hits the point a in Fig. 20a, $x \rightarrow 1$. Now we follow pictures Fig. 20 alphabetically, in the direction of increasing x . So in Fig. 20a the slit shortens. As it vanishes we obtain Fig. 20b, transformation 1 happens, and we pass to Fig. 20c.

In Fig. 20c, the slit lengthens till q hits b . A transformation 2 happens, detaching a digon with corners x and b in Fig. 20c to obtain the quadrilateral Fig. 20d. A point c in Fig. 20d maps to the same point as x . A digon with the corners b and c is attached to the interval $[b, c]$ of the quadrilateral, resulting in a pentagon Fig. 20e. The slit shortens towards x in Fig. 20e. When it hits x , the points x and c collide, and we get a quadrilateral in Fig. 20f. A transformation 1 happens at Fig. 20f, and the slit lengthens towards 1 in Fig. 20g. As it hits 1, a disk with the red (dotted line) boundary is detached, resulting in the quadrilateral Fig. 20h. The point e in Fig. 20h maps to the same point as 1. A transformation 3 happens in Fig. 20h, with a disk with black (solid line) boundary attached in Fig. 20i. As the slit in Fig. 20i shortens and q hits 1, the points 1 and e collide, and we get the quadrilateral Fig. 20j.

Transformations occurring in the right column of Fig. 20 are: 1,2,1,3,1. The last quadrilateral shown is Fig. 20j. It is symmetric with respect to the reflection which exchanges 0 and x while leaving 1 and ∞ fixed. It is easy to see that in the further continuation of the process we will obtain all pictures Fig. 20 in the reverse order i)-a) subject to a reflection exchanging 0 and x .

So Fig. 20 represents only one half of the global family. The global family is symmetric, with the symmetry exchanging (0) and (x). The full sequence of special points is

$$(1, x, x, 1, 1, 1, 0, 0, 1).$$

Appendix I. Monodromy representations corresponding to quadrilaterals

A monodromy representation consists of 4 matrices in $SL(2, \mathbf{C})$ which satisfy the relation (10). For real equations (4) these four matrices can be represented as products of reflections in the circles C_j containing the images of the sides of a special pentagon. Here we will discuss which monodromy representations correspond to real equations, and how to find the reflections σ_j from matrices T_j .

This problem was addressed in [7], and we begin by restating the result

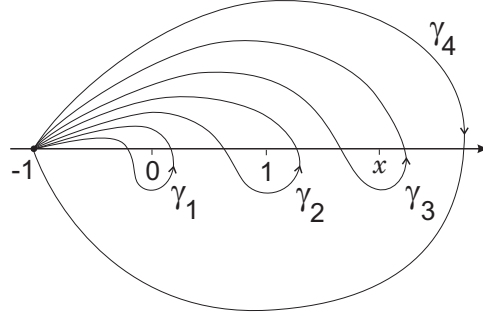


Figure 21: Modified loops.

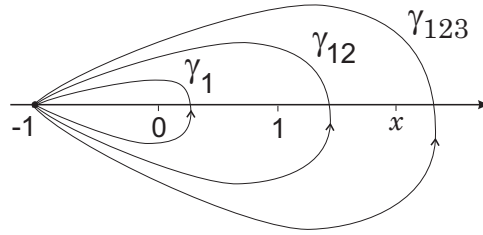


Figure 22: Symmetric loops.

obtained there. First of all, we change the reference point of the fundamental group in Fig. 1 to the point -1 as in Fig. 21, and deform the loops accordingly. Now consider a symmetric set of generators of the fundamental group shown in Fig. 22. Let N_1, N_2, N_3 be the monodromy matrices corresponding to $\gamma_1, \gamma_{12}, \gamma_{123}$. We have

$$N_1 = T_1, \quad N_2 = T_1 T_2, \quad N_3 = T_1 T_2 T_3.$$

When none of the κ_j is an integer, monodromy representation determines equation (4) uniquely for given real x and κ_j [7, 4.2, 4.3]. This implies that monodromy representations correspond to real solutions of (4) normalized as in (9) with $x_0 = -1$ if and only if

$$\overline{N_j} = N_j^{-1}, \quad 1 \leq j \leq 3, \quad (32)$$

which is equivalent to the condition obtained in [7].

We will derive a different condition, without the assumption on the κ_j .

Consider an arbitrary quadruple of $SL(2, \mathbf{C})$ matrices satisfying

$$T_1 T_2 T_3 T_4 = \text{id}. \quad (33)$$

$SL(2, \mathbf{C})$ acts on these quadruples by simultaneous conjugation. To parametrize conjugacy classes of monodromy representations we denote

$$t_j = \text{Tr } T_j, \quad t_{jk} = \text{Tr } (T_j T_k) = \text{Tr } (T_k T_j).$$

Conjugacy classes are parametrized by 7 complex numbers

$$t_1, t_2, t_3, t_4, t_{12}, t_{23}, t_{13} \quad (34)$$

which are subject to one relation

$$\begin{aligned} & t_{12}t_{23}t_{13} + t_{12}^2 + t_{23}^2 + t_{13}^2 \\ & - t_{12}(t_1t_2 + t_3t_4) - t_{23}(t_2t_3 + t_1t_4) - t_{13}(t_1t_3 + t_2t_4) \\ & + t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_1t_2t_3t_4 = 4. \end{aligned} \quad (35)$$

This relation was found by Fricke and Klein [15], and was studied in [23], [3] and elsewhere. Parametrization of monodromy representations by these data is discussed in detail in [27]. In particular it is proved there that there are open dense sets on the hypersurface (35) and on the space of conjugacy classes of monodromy representations which are homeomorphic.

We say that a representation is *generated by reflections* if there exist four circles C_1, C_2, C_3, C_4 such that the reflections σ_j in these circles satisfy

$$T_j = \sigma_j \sigma_{j+1}, \quad j \in \mathbf{Z}_4. \quad (36)$$

Notice that (36) implies (32).

Arbitrary reflection can be written as

$$\sigma(z) = \frac{a\bar{z} + b}{c\bar{z} - \bar{a}}, \quad (37)$$

which we represent by the matrix

$$\begin{pmatrix} a & b \\ c & -\bar{a} \end{pmatrix}, \quad |a|^2 + bc = 1, \quad (38)$$

where b, c are real. Product of reflections represented by matrices A, B is a linear-fractional transformation with matrix $A\bar{B}$. Matrices A associated with reflections are characterized by the properties that $\det A = -1$ and $\bar{A} = A^{-1}$.

Let Σ_j be the matrices representing the reflections σ_j . Then $\Sigma_1 = I$ because of our normalization, and we have $T_1 = \bar{\Sigma}_2$, $T_2 = \Sigma_2 \bar{\Sigma}_3$, $T_3 = \Sigma_3 \bar{\Sigma}_4$.

So $N_1 = T_1$ has matrix $\overline{\Sigma_2}$, $N_2 = T_1 T_2$ has matrix $\overline{\Sigma_2 \Sigma_2 \Sigma_3} = \overline{\Sigma_3}$, and $N_3 = T_1 T_2 T_3$ has matrix $\overline{\Sigma_3 \Sigma_3 \Sigma_4} = \overline{\Sigma_4}$. Thus (32) is satisfied.

Our first question is which representations are generated by reflections.

First we notice that composition of two reflections always has real trace: it is elliptic if the circles cross, parabolic if they are tangent and hyperbolic if they are disjoint. Second, if (36) holds then $T_j T_{j+1} = \sigma_j \sigma_{j+2}$ also has real trace for each $j \in \mathbf{Z}_4$. Thus if (36) holds, the first six parameters in (34) must be real. In addition to this we have the following inequality:

Theorem A1. *A monodromy representation (33) is generated by reflections if and only if $t_1, t_2, t_3, t_4, t_{12}, t_{23}$ are real and*

$$\begin{aligned} \Delta &:= t_1^2 t_3^2 + t_2^2 t_4^2 + t_{12}^2 t_{23}^2 - 4(t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_{12}^2 + t_{23}^2) \\ &\quad + 4(t_1 t_2 t_{12} + t_2 t_3 t_{23} + t_1 t_4 t_{23} + t_3 t_4 t_{12}) \\ &\quad - 2(t_1 t_2 t_3 t_4 + t_2 t_4 t_{12} t_{23} + t_1 t_3 t_{12} t_{23}) + 16 \leq 0. \end{aligned} \quad (39)$$

Monodromy transformations T_j determine the reflections σ_j uniquely unless all T_j commute, and the projective monodromy group is isomorphic to a subgroup of the multiplicative group of the unit circle or of the additive group of the real line.

Proof of Theorem A1.

Uniqueness. Suppose that we have (36) and

$$T_j = \sigma'_j \sigma'_{j+1}, \quad j \in \mathbf{Z}_4. \quad (40)$$

First we notice that if $\sigma_j = \sigma'_j$ for some j , then $\sigma_k = \sigma'_k$ for all k . Indeed $\sigma_j = \sigma'_j$ together with (36) and (40) implies $\sigma_{j+1} = \sigma'_{j+1}$ and so on.

Therefore, it is sufficient to prove that $\sigma_2 = \sigma'_2$. We have

$$T_1 = \sigma_1 \sigma_2, \quad T_2 = \sigma_2 \sigma_3. \quad (41)$$

Lemma A1. *If T_1 and T_2 are non-identical linear-fractional transformations which together have at least three fixed points, and (41) holds, then σ_2 is the reflection in the unique circle which passes through all fixed points of T_1 and T_2 .*

Proof. If $T_1 \neq \text{id}$ then the circles C_1 and C_2 of σ_1 and σ_2 are distinct and their points of intersection are exactly the fixed points of T_1 . So C_2 contains the fixed points of T_1 , and T_2 . This proves the lemma.

How can T_1 and T_2 have at most 2 fixed points together?

- a) One is elliptic and another one is parabolic, sharing one fixed point.
- b) Both are parabolic.
- c) Both are elliptic sharing two fixed points, in which case they commute.

Consider the case a). Suppose that the shared fixed point is ∞ , $T_1(z) = e^{2\pi i\alpha}z$ and $T_2(z) = z + c$. Then C_1 and C_2 must be lines through the origin, and C_2, C_3 must be parallel lines perpendicular to c . Therefore C_2 is the line through the origin perpendicular to c , that is this circle is uniquely defined by T_1 and T_2 .

Now we address b). If in case b) T_1 and T_2 do not share their fixed points, we may assume that $T_1(z) = z + c$ while T_2 has fixed point $d \in \mathbf{C}$. Then C_2 is the unique line through d perpendicular to c .

If the parabolic transformations in b) share the fixed point, then they are simultaneously conjugate to $z + a$ and $z + b$, and C_2 is a line perpendicular to both a and b , so a and b are collinear.

So either the circle C_2 is uniquely defined by T_1, T_2 , or T_1 and T_2 commute, and either both are elliptic or both are parabolic. If they are both parabolic, their families of invariant circles must be the same.

This argument applies to every pair T_k, T_{k+1} . Therefore, the only cases when the σ_j are not defined by the T_j are the cases stated in the theorem. This completes the proof of uniqueness.

Existence. We have already noticed that reality of $t_1 \dots, t_4, t_{12}, t_{23}$ is necessary for (36). It remains to prove that when these traces are real, inequality (39) is necessary and sufficient.

We write a reflection as in (37), (38) In particular, we obtain the reflection in the real axis when $a = i, b = c = 0$, and in the line $e^{i\alpha}$ when $a = ie^{i\alpha}, b = c = 0$.

The trace of a product is

$$\text{tr}(\sigma_1\sigma_2) = 2\text{Re}(a_1\overline{a_2}) + b_1c_2 + c_1b_2. \quad (42)$$

We normalize by $SU(2)$ conjugation so that two adjacent circles are the real line and the line $\{re^{i\alpha} : r \in \mathbf{R}\}$, and write the four matrices of reflections that we want to find as

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & -\overline{a_1} \end{pmatrix}, \quad \begin{pmatrix} ie^{i\alpha} & 0 \\ 0 & ie^{-i\alpha} \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \begin{pmatrix} a_2 & b_2 \\ c_2 & -\overline{a_2} \end{pmatrix},$$

in this order. We can further normalize, and assume that 1 is a fixed point of the product of the third and fourth reflections:

$$\frac{i\overline{a_2}z + ib_2}{ic_2z - ia_2} = z, \quad \text{where } z = 1, \quad (43)$$

which gives

$$c_2 - b_2 = 2\operatorname{Re} a_2. \quad (44)$$

Now we write that the traces of products are given (real) numbers:

$$2\operatorname{Re}(-ie^{-i\alpha}a_1) = t_1, \quad (45)$$

$$2\cos\alpha = t_2, \quad (46)$$

$$2\operatorname{Re}(i\overline{a_2}) = t_3, \quad (47)$$

$$2\operatorname{Re}(a_2\overline{a_1}) + b_2c_1 + c_2b_1 = t_4, \quad (48)$$

$$2\operatorname{Re}(-ia_1) = t_{12}, \quad (49)$$

$$2\operatorname{Re}(ie^{i\alpha}\overline{a_2}) = t_{23}. \quad (50)$$

Equations (44)–(50) are easy to solve. First, a_1 is determined from (45), (49) and a_2 from (47), (50). Then products b_jc_j are found from

$$b_1c_1 = 1 - |a_1|^2, \quad b_2c_2 = 1 - |a_2|^2, \quad (51)$$

which express the fact that determinants of our matrices are -1 , and together with (44) and (48) permits to find b_j, c_j . This amounts to solving two quadratic equations. One of them always has real solutions. Inequality (39) comes from the condition that the second also has real solutions, namely that all b_j, c_j are real.

We give the details of the computation. From (45), (49),

$$-ia_1 = \frac{1}{2} \left(t_{12} - i \frac{t_{12} \cos \alpha - t_1}{\sin \alpha} \right). \quad (52)$$

Similarly, from (47), (50),

$$i\overline{a_2} = \frac{1}{2} \left(t_3 + i \frac{t_3 \cos \alpha - t_{23}}{\sin \alpha} \right). \quad (53)$$

Then

$$1 - |a_1|^2 = \frac{1}{t_2^2 - 4} (t_1^2 + t_2^2 + t_{12}^2 - t_1 t_2 t_{12} - 4), \quad (54)$$

and

$$1 - |a_2|^2 = \frac{1}{t_2^2 - 4} (t_2^2 + t_3^2 + t_{23}^2 - t_2 t_3 t_{23} - 4). \quad (55)$$

Using (52) and (53) we obtain

$$2\operatorname{Re}(a_2 \overline{a_1}) = \frac{1}{4 - t_2^2} (2t_1 t_{23} + 2t_3 t_{12} - t_1 t_2 t_3 - t_2 t_{12} t_{23}), \quad (56)$$

and using (44)

$$c_2 - b_2 = \frac{t_3 \cos \alpha - t_{23}}{\sin \alpha}. \quad (57)$$

Next, from (51), (54), (55), we obtain

$$b_1 c_1 = \frac{1}{t_2^2 - 4} (t_1^2 + t_2^2 + t_{12}^2 - t_1 t_2 t_{12} - 4), \quad (58)$$

and

$$b_2 c_2 = \frac{1}{t_2^2 - 4} (t_2^2 + t_3^2 + t_{23}^2 - t_2 t_3 t_{23} - 4). \quad (59)$$

Solving first the system (57), (59) with respect to b_2, c_2 , we obtain a quadratic equation with discriminant

$$t_2^2 t_3^2 - 4t_2^2 - 4t_3^2 + 16 = (t_2^2 - 4)(t_3^2 - 4) \geq 0,$$

because $t_j \in [-2, 2]$. So we always have real solution c_2, b_2 .

Next we solve the system (48) with (56) and (58) with respect to b_1, c_1 , using the known product $b_2 c_2$ from (59). This also leads to a quadratic equation, whose discriminant is a polynomial in t_{12}, t_{23} and t_j . This polynomial factors (using Maple) with one factor $t_2^2 - 4 < 0$ and the other factor is Δ in (39).

This completes the proof.

Remark on the proof. In the process of recovery of σ_j we had to solve two quadratic equations, so in general we had 4 choices to make. On the other hand, our normalization condition (43) leaves two choices because two circles intersect at two points. Next, we never used t_{13} in our recovery procedure

for σ_j . As t_{13} satisfies the quadratic equation (35), assigning t_{13} narrows our choices to two.

An interesting question is what happens when the monodromy group is conjugate to a subgroup of $SU(2)$. Every element of $SU(2)$ is the product of two reflections in great circles. If an element of $SU(2)$ is represented as a product of two reflections, then these reflections must be in great circles, because these circles contain the fixed points of the element which are diametrically opposite.

An interesting special case is when all seven parameters in (34) are real. According to [34, Prop. III.1.1] this happens if and only if the projective monodromy group is a subgroup of $SU(2)$ or $SL(2, \mathbf{R})$. When the group is generated by reflections, the first six parameters in (34) are real, so all seven will be real if and only if the discriminant of (35), as a quadratic equation with respect to t_{13} , is non-negative. Straightforward computation shows that this discriminant is nothing but Δ defined in (39). Thus we obtain

Theorem A2. *Let T_1, T_2, T_3, T_4 be unitary matrices satisfying (33), and all seven parameters in (34) are real. This representation is generated by reflections if and only if $\Delta = 0$, which is equivalent to*

$$2t_{13} + t_{12}t_{23} - t_1t_3 - t_2t_4 = 0. \quad (60)$$

Equation (60) is what (35) becomes when $\Delta = 0$.

Remarks. We mention a simple geometric interpretation of our conditions.

Condition that t_{12}, t_{23} are real: the trace of a product of two elliptic transformations is real if and only if their four fixed points lie on a circle.

Condition $\Delta = 0$ gives a relation between six angles associated to a spherical or hyperbolic quadrilateral: four angles of the quadrilateral, and two angles between the circles containing the images of opposite sides. These six angles serve as natural parameters: spherical or hyperbolic quadrilaterals with prescribed angles at the corners form a one-parametric family, while circular quadrilaterals with prescribed corners form a two-parametric family.

Theorem A2 can be compared with Jimbo's asymptotics [28]. (A misprint in the main result in [28] was corrected in [4]). It follows from the explicit formula expressing the asymptotics in terms of the monodromy that for $SU(2)$ or $SL(2, \mathbf{R})$ monodromies this asymptotics is real if and only if

$\Delta = 0$. For general monodromies (not in $SU(2)$) it is difficult to determine directly when Jimbo's formula gives a real asymptotics.

Appendix II. Topological classification of 4-circle chains.

We recall that a 4-circle chain consists of 4 labeled circles C_j on the Riemann sphere such that

$$C_j \neq C_{j+1}, \quad C_j \cap C_{j+1} \neq \emptyset, \quad j \in \mathbf{Z}_4. \quad (61)$$

In this section we give a topological classification of generic chains. Generic means that there are no tangent circles and no triple intersections. Two chains are considered equivalent if there is an orientation-preserving homeomorphism of the sphere which maps the union of circles of one chain onto the union of circles of another chain.

In fact, we classify generic *unordered* quadruples of circles with the following property: each circle intersects at least two other circles. There are two kinds of such quadruples: those in which each circle intersects all three other circles (see Fig. 23) and those with a pair of non-intersecting circles (see Fig. 24). Note that the quadruples in Fig. 23D and 24K are not reflection symmetric, thus each of them represents two equivalence classes. We omit the elementary but tedious proof that these exhaust all possibilities. To see that all these cell decompositions are distinct we indicated the faces with more than 3 edges in each cell decomposition.

It follows from the classification that the circles in Fig. 23 can be arbitrarily ordered to form a 4-circle chain, while the circles in Fig. 24 form a 4-circle chain when ordered so that non-intersecting circles are not adjacent.

Notice two equivalent reformulations of this problem: topological classification of arrangements of four planes in the hyperbolic space, subject to the intersection condition (61), and topological classification of possible intersections of a sphere with four planes in the Euclidean space, under the condition that the planes P_j can be so ordered that each line $P_j \cap P_{j+1}$ intersects the sphere.

Remarks and conjectures.

If all pairs of circles in the chain intersect, then there are only finitely many nets on this chain with prescribed angles. This follows from the results of [25]. In this paper, Ihlenburg proves that all circular quadrilaterals can be obtained from finitely many topological types by applying four explicitly

defined operations. All these operations do not decrease angles, and three of them increase some angles. The only operation which leaves all angles unchanged requires two disjoint circles in the chain.

It follows that real solutions of PVI corresponding to all chains in Fig. 23 can have only finitely many special points on an interval between fixed singularities. On the other hand, our examples 4, 6 suggest that for all chains containing pairs of disjoint circles the number of special points is infinite. Moreover, it looks like it is infinite in one direction when there is one pair of disjoint circles, and infinite in both directions if there are two such pairs, like in Fig.18 which is the same as Fig. 24O.

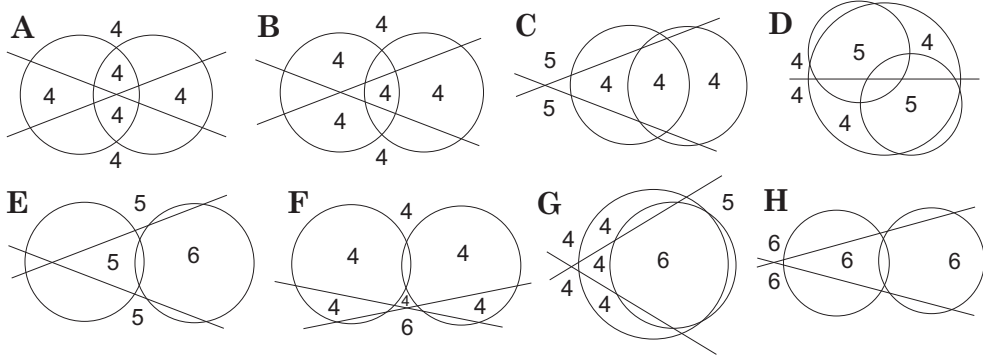


Figure 23: Generic chains A-H. All pairs of circles intersect.

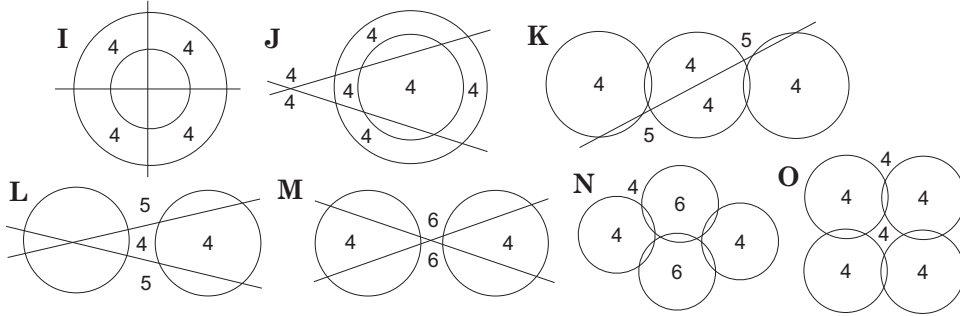


Figure 24: Generic chains I-O. Some pairs are disjoint.

Appendix III. Moduli of conformal quadrilaterals.

Consider a closed rectangle Q in the plane with vertices $0, 1, 1 + ia, ia$. The number $a > 0$ is called the modulus, $a = \text{mod } Q$. Any Borel measurable

function $\rho(z) \geq 0$ defines a *conformal metric* $\rho(z)|dz|$ on Q : the length of a curve γ and the area of a set $E \subset Q$ are defined as

$$\ell_\rho(\gamma) = \int_\gamma \rho(z)|dz|, \quad \text{and} \quad A_\rho(E) = \int_E \rho^2(z) dx dy.$$

Let Γ be the set of all curves in Q connecting the horizontal sides. Define

$$\ell_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \ell_\rho(\gamma),$$

and

$$\lambda(\Gamma) = \sup_\rho \frac{\ell_\rho^2(\Gamma)}{A_\rho(Q)}, \quad (62)$$

where the sup is taken over all metrics for which the numerator and denominator are finite and not zero.

Lemma A2. [2, I.D, Example 1] $\lambda(\Gamma) = a$.

Formula (62) defines the *extremal length* of an *arbitrary* family of curves Γ in Q . So defined extremal length is a conformal invariant of a family of curves.

Let Γ' be the family of all curves in Q connecting the vertical sides. Then evidently

$$\lambda(\Gamma)\lambda(\Gamma') = 1. \quad (63)$$

The following comparison inequalities immediately follow the from definition.

Lemma A3. [2, I.D, Theorem 2] *Consider two families of curves Γ_1 and Γ_2 and suppose that every curve $\gamma_2 \in \Gamma_2$ contains some curve $\gamma_1 \in \Gamma_1$. Then $\lambda(\Gamma_1) \leq \lambda(\Gamma_2)$.*

The assumption means that Γ_1 has “more curves” and the curves of Γ_2 are “longer”.

For a metric ρ , the intrinsic distance $d_\rho(E_1, E_2)$ between two subsets E_1 and E_2 of Q is defined as infimum of $\ell_\rho(\gamma)$ over all curves connecting a point in E_1 with a point in E_2 .

Lemma A4. *Suppose that a metric ρ has the following properties:*

$$A_\rho(B(r)) \leq Kr^2, \quad (64)$$

for all $r > 0$ and for all intrinsic disks $B(r)$ of radii r , the intrinsic ρ -distance between the vertical sides is at least $2c$, and the intrinsic ρ -distance between the horizontal sides is less than $\epsilon/2$.

Then $\text{mod } Q \leq \delta$, where

$$\delta = \frac{4(K+1)}{\log(c/\epsilon)} \rightarrow 0$$

as $\epsilon \rightarrow 0$, for any fixed $K > 0, c > 0$.

Proof. Choose a curve γ_0 connecting the horizontal sides, and such that $\ell_\rho(\gamma_0) < \epsilon$. Let P be a point on γ_0 . Consider the closed ρ -disks $B(\epsilon)$ and $B(c)$ of radii ϵ and c , both centered at P . Then $B(\epsilon)$ contains γ_0 . Let Γ' be the family of all curves in Q connecting the vertical sides. Every curve γ' of this family crosses γ_0 , therefore γ' intersects both $B(\epsilon)$ and $Q \setminus B(c)$. Therefore

$$\lambda(\Gamma') \geq \lambda(\Gamma_1), \quad (65)$$

where Γ_1 is the family of all curves in Q connecting $B(\epsilon)$ with $Q \setminus B(c)$. To estimate $\lambda(\Gamma_1)$ from below, consider the metric $\tau(z)|dz|$ defined by the function

$$\tau(z) = \frac{\rho(z)}{\log(c/\epsilon)d_\rho(z, P)}, \quad z \in B(c) \setminus B(\epsilon),$$

and zero otherwise. For every $\gamma_1 \in \Gamma_1$ we have

$$\ell_\tau(\gamma_1) \geq \frac{1}{\log(c/\epsilon)} \int_\epsilon^c \frac{\rho(z)|dz|}{d_\rho(z, P)} \geq \frac{1}{\log(c/\epsilon)} \int_\epsilon^c \frac{ds}{s} \geq 1.$$

Here we made the change of the variable $s = d_\rho(z, P)$ and used the evident inequality $ds \leq \rho(z)|dz|$.

To estimate the τ -area of Q we define $r_0 = \epsilon$, $r_k = 2^k r_0$, $k = 1, \dots, N$, $N = \lceil \log_2(c/\epsilon) \rceil + 1$, and let B_k be the ρ -disk of radius r_k centered at P . Then, using (64), we obtain

$$\begin{aligned} A_\rho(Q) &\leq \frac{1}{\log^2(c/\epsilon)} \sum_{k=1}^N \int_{B_k \setminus B_{k-1}} \frac{\rho^2(z) dx dy}{r_{k-1}^2} \leq \frac{4}{\log^2(c/\epsilon)} \sum_{k=1}^N \frac{A_\rho(B_k)}{r_k^2} \\ &\leq \frac{4KN}{\log^2(c/\epsilon)} \leq \frac{4(K+1)}{\log(c/\epsilon)}. \end{aligned}$$

Thus $\lambda(\Gamma_1) \geq (\log(c/\epsilon))/(4(K+1))$, and using (63) and (65), we obtain

$$\text{mod } Q = \lambda(\Gamma) \leq 4(K+1)/(\log(c/\epsilon)).$$

This proves the lemma.

The upper half-plane can be mapped conformally onto Q so that

$$(0, 1, x, \infty) \mapsto (0, 1, 1 + ia, ia)$$

by the Schwarz–Christoffel formula. Let

$$\phi(z) = \int_0^z \frac{d\zeta}{\sqrt{z(\zeta - 1)(\zeta - x)}}.$$

Then the desired conformal map is $\phi(z)/\phi(1)$, and the modulus $a(x) = -i\phi(\infty)/\phi(1)$. It follows from Lemma A3 that $x \mapsto a(x)$ is increasing homeomorphism of $(1, \infty)$ onto $(0, +\infty)$.

In our applications, the metric ρ arises as a pull-back of the standard spherical metric of curvature 1 on the sphere S by a conformal local homeomorphism $f : Q \rightarrow S$. If f is p -valent (which means that every point has at most p preimages), then (64) is satisfied with $K = \pi p$. Indeed, for spherical discs on S , (64) is satisfied with $K = \pi$ by direct computation, and $f(B(r))$ is evidently contained in a disk of radius r in S .

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