Uniform approximation of $\operatorname{sgn} x$ by polynomials and entire functions

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In 1877, E. I. Zolotarev [19, 2] found an explicit expression, in terms of elliptic functions, of the rational function of given degree m which is uniformly closest to $\operatorname{sgn}(x)$ on the union of two intervals $[-1, -a] \cup [a, 1]$. This result was subject to many generalizations, and it has applications in electric engineering.

Surprisingly, to the best of our knowledge, the similar problem for polynomials was not solved yet, so we investigate it in this paper.

For comparison, we mention here the results on the uniform approximation of $|x|^\alpha,\ \alpha > 0$ on $[-1, 1]$. Polynomial approximation was studied by S. Bernstein [3, 4] who found that for the error $E_m(\alpha)$ of the best approximation by polynomials of degree m the following limit exists:

$$\lim_{m\to\infty} m^\alpha E_m(\alpha) = \mu(\alpha) > 0.$$

This result for $\alpha = 1$ was obtained by Bernstein in 1914, and he asked the question, whether one can express $\mu(1)$ in terms of some known transcendental functions. This question is still open. Bernstein also obtained in [4] the asymptotic relation

$$\lim_{\alpha\to 0} \mu(\alpha) = 1/2.$$

The analogous problem of uniform rational approximation of $|x|^\alpha,\ \alpha > 0$ on $[-1, 1]$ was recently solved by H. Stahl [14], who completed a long line of

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development with a remarkably explicit answer:

$$\lim_{m\to\infty} \exp(\pi\sqrt{\alpha m})E^r_m = 2^{2+\alpha}|\sin(\pi\alpha/2)|,$$

where E^r_m is the error of the best rational approximation.

Now we state our results. Let p_m be the polynomial of degree at most $2m+1$ of least deviation from $\operatorname{sgn}(x)$ on $X(a) = [-1,-a] \cup [a,1]$ where $0 < a < 1$. It follows from the general theory of Chebyshev that such polynomial is unique. Put $L_m(a) = \max_{X(a)} |p_m(x) - \operatorname{sgn}(x)|$. Then we have

Theorem 1 *The following limit exists*

$$\lim_{m\to\infty} \sqrt{m}\left(\frac{1+a}{1-a}\right)^m L_m(a) = \frac{1-a}{\sqrt{\pi a}}.$$

Remark. When approximating an odd function on a symmetric set, polynomials of even degrees are useless. Indeed, if q is a polynomial of degree at most $2m$ which deviates least from our function, then $(q(z)-q(-z))/2$ is an odd polynomial, and its deviation is at most that of q. Thus q has to be of odd degree.

Our approximation problem is equivalent to a problem of weighted approximation on a single interval. Indeed, p_m can be written as $p_m(x) = xq_m(x^2)$, where q_m is the polynomial of degree m that minimizes the weighted uniform distance

$$\sup_{[a^2,1]} \sqrt{x}|q(x) - 1/\sqrt{x}|. \tag{1}$$

over all polynomials q of degree at most m.

It is useful to compare our result with the result of Bernstein [6], see also [1, Additions and Problems, 44], that gives the rate of the best *unweighted* uniform polynomial approximation of $1/\sqrt{x}$:

$$\lim_{m\to\infty} \sqrt{m}\left(\frac{1+a}{1-a}\right)^m \inf_{\deg r=m} \sup_{[a^2,1]} |r(x) - 1/\sqrt{x}| = \frac{1}{2\sqrt{\pi}}(1-a^2)a^{-3/2}. \tag{2}$$

In our proof of Theorem 1, the asymptotics of the error term is obtained in the form

$$\lim_{m\to\infty} \sqrt{m}\left(\frac{1+a}{1-a}\right)^m L_m(a) = \frac{\sqrt{2}(1-a)}{\sqrt{a}}e^{-c}, \tag{3}$$

with

$$c = \frac{1}{\pi} \int_0^\infty \left(\Im H(t) - \frac{\pi}{2} \chi_{[2,\infty)}(t) \right) \frac{dt}{t}, \tag{4}$$

where $\chi_{[2,\infty)}$ is the characteristic function of the ray $[2, \infty)$, and H is the conformal map of the upper half-plane onto the region in the upper half plane above the curve

$$\{t + i \arccos e^{-t} : t \geq 0\} \cup \{t : t < 0\},$$

normalized by $H(0) = 0$ and $H(z) \sim z$, $z \to \infty$. Then the numerical value $c = (1/2) \log(2\pi)$ is derived from comparison of (3) with (2) for $a \to 1$. So, as a curious corollary from (3) and the result of Bernstein (2), we evaluate the integral (4).

Following Bernstein, we also consider approximation by entire functions of exponential type. Let $L(A)$ be the error of the best uniform approximation of $\operatorname{sgn}(x)$ by entire functions of exponential type one, on the set $(-\infty, -A] \cup [A, +\infty)$.

Theorem 2 *The following limit exists*

$$\lim_{A \to \infty} \sqrt{A} \exp(A) L(A) = \sqrt{2/\pi}.$$

The proof of Theorem 2 is similar to (and simpler than) that of Theorem 1. On the best L^1 approximation of $\operatorname{sgn}(x)$ by entire functions of exponential type we refer to [16].

Our proofs are based on special representations of polynomials and entire functions of best approximation which are of independent interest. It follows from Chebyshev's theory (see, for example, [1, Ch. II]) that polynomials p_m are characterized by the property that the difference $p_m(x) - \operatorname{sgn}(x)$ takes its extreme values $\pm L_m(a)$ on $X(a)$ $2m + 4$ times intermittently, so that the graph of p_m looks like this:

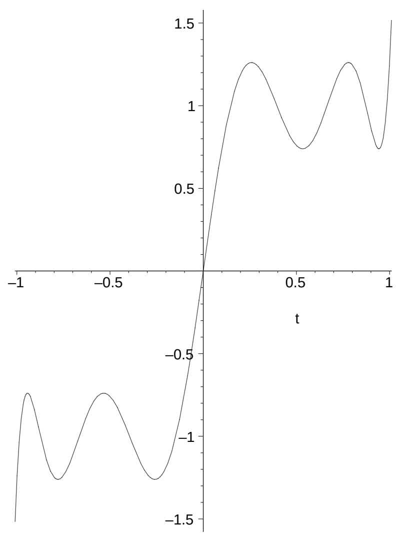

Fig. 1. Graph of p_4 with $a = 0.1$.

The extremal entire function is unique and is characterized by the properties that it has no asymptotic values, all its critical values are real; those on the negative ray are $-1 \pm L(A)$ and those on the positive ray are $1 \pm L(A)$.

We mention a general theorem of Maclane [11] and Vinberg [18] on the existence and uniqueness of real polynomials and entire functions with prescribed (ordered!) sequences of critical values. In the case of polynomials, this theorem says that there is a one to one correspondence between finite "up-down" real sequences

$$\ldots \leq c_{k-1} \geq c_k \leq c_{k+1} \geq \ldots,$$

and real polynomials whose all critical points are real, modulo a change of the independent variable $z \mapsto az + b,\ a > 0, b \in \mathbf{R}$. If (x_k) is the sequence of critical points of such polynomial, then $c_k = P(x_k)$.

The MacLane–Vinberg theorem is based on an explicit description of the Riemann surfaces spread over the plane of the inverse functions P^{-1}. This explicit description and our previous work [7], [15] suggest to look for a representation of these extremal polynomials and entire functions in the form $\cos \phi(z)$ where ϕ is an appropriate conformal map.

Let L and B be positive numbers, $0 < L < 1$, and

$$L = \frac{1}{\cosh B} \sim 2e^{-B}, \quad B \to \infty. \tag{5}$$

Consider the component $\gamma_B \subset \{z : \Re z \in [0, \pi], \Im z > 0\}$ of the preimage of the ray

$$\{w : \Re w = 1/L,\ \Im w < 0\}$$

under $w = \cos z$. It is easy to see that this curve γ_B can be parametrized as

$$\gamma_B = \{\arccos(\cosh B / \cosh t) + it : B \leq t < \infty\}.$$

This curve begins at iB and then goes to infinity approaching the line $\{\pi/2 + it, t > 0\}$ with exponential rate.

Let Ω_B be the region in the upper half-plane whose boundary consists of the positive ray, the vertical segment $[0, iB]$ and the curve γ_B. For fixed $B > 0$, let ϕ_B be the conformal map of the first quadrant onto Ω_B such that $\phi_B(z) \sim z,\ z \to \infty$ and $\phi_B(0) = iB$. Let $A = A(B) = \phi_B^{-1}(0)$. Then A is a continuous strictly increasing function of B, and we may consider the inverse function $B(A)$.

Theorem 3 *The approximation error in Theorem 2 is $L(A) = 1/\cosh B(A)$, and the extremal function can be defined in the first quadrant by the formula*

$$1 - L(A) \cos \phi_{B(A)}.$$

Let $\Omega_{B,m}$ be the region in the half-strip $\{z : \Re z \in (0, \pi(m+1)), \Im z > 0\}$ bounded on the left by γ_B. Let $\phi_{B,m}$ be the conformal map of the first quadrant onto $\Omega_{B,m}$ such that $\phi_{B,m}(0) = iB,\ \phi_{B,m}(1) = \pi(m+1)$ and $\phi_{B,m}(\infty) = \infty$. Let $a = a(B, m) = \phi_{B,m}^{-1}(0)$. Then $a(B, m)$ is a continuous increasing function of B for fixed m, so it has the inverse $B_m(a)$.

Theorem 4 *The error term in Theorem 1 is $L_m(a) = 1/\cosh B_m(a)$, and the extremal polynomial is given in the first quadrant by*

$$1 - L_m(a) \cos \phi_{B,m}, \quad \textit{where} \quad B = B_m(a).$$

Discontinuous functions cannot be uniformly approximated by polynomials with arbitrarily small precision, however, in our situation we can obtain an approximation which seems to be the second best thing to the uniform approximation.

We introduce the notation[1]

$$\mathcal{L}(f,g)=\inf\{h: f(x-h)-h\leq g(x)\leq f(x+h)+h,\ -1\leq x\leq 1\}.$$

Thus the statement that $\mathcal{L}(\mathrm{sgn}\,,g)\leq\epsilon$ means that the graph of the restriction of g on $[-1,1]$ belongs to a "rectangular corridor" of width ϵ around the "completed graph" of $\mathrm{sgn}\,(x)$, which consists of the graph of $\mathrm{sgn}\,(x)$ and the vertical segment $[(0,-1),(0,1)]$.

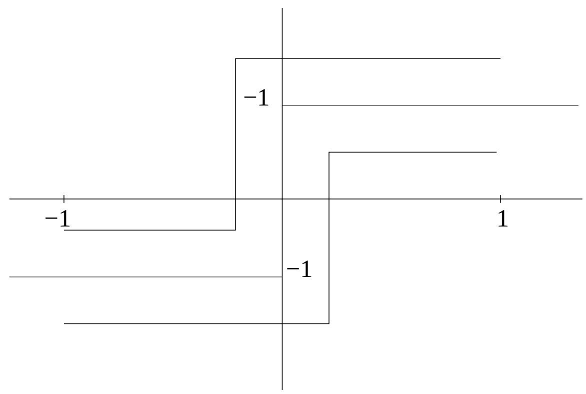

Fig. 2. Lévy's neighborhood of the function sgn .

It is easy to see that if $a=L_m(a)$ then our polynomial p_m from Theorem 1 is the unique polynomial of degree $2m+1$ which minimizes $\mathcal{L}(\mathrm{sgn}\,,p)$. We have

Theorem 5

$$\lim_{m\to\infty}\frac{m}{\log m}\mathcal{L}(\mathrm{sgn}\,,p_m)=\frac{1}{2}.$$

Remarks. One could also use the Hausdorff distance between the completed graphs. In the case of $\mathrm{sgn}\,(x)$, the Hausdorff distance will differ from $\mathcal{L}(\mathrm{sgn}\,,p_m)$ by a factor of $\sqrt{2}$. Approximation of functions with respect to Hausdorff distance between their completed graphs was much studied by Sendov [12] and his followers. An *arbitrary* bounded function on $[0,1]$ can be approximated in this sense by polynomials of degree n with error $O((\log n)/n)$ [13].

[1] If $[-1,1]$ is replaced by $(-\infty,\infty)$ this becomes the Lévy distance. It is really a distance on the set of bounded increasing functions on the real line [10, Ch. VIII].

Proof of Theorems 3 and 4. Let ϕ be either ϕ_B or $\phi_{B,m}$. By inspection of the boundary correspondence, we conclude that $f = 1 - L\cos\phi$ is real on the positive ray and pure imaginary on the positive imaginary ray. So f extends to an entire function by two reflections. The extended function evidently satisfies

$$\overline{f(\overline{z})} = f(z) \quad \text{and} \quad -\overline{f(-\overline{z})} = f(z),$$

so we conclude that f is odd. In the case of Theorem 1, the region $\Omega_{B,m}$ is close to the strip $\{z : \Re z \in (\pi/2, \pi(m+1))\}$ as $\Im z \to \infty$, so $\phi_{B,m} \sim i(2m+1)\log z,\ z \to \infty$, so f is a polynomial of degree $2m+1$. In Theorem 2, $\phi(z) \sim z$, so f has exponential type one. In both theorems, differentiation shows that the only critical points of f in the closed right half-plane are preimages of the critical points of the cosine under ϕ. So the graph of f has the required shape. That a polynomial with such graph is the unique extremal for Theorem 1 follows from the general theorem of Chebyshev on the uniform approximation of continuous functions [1, Ch. II].

The proof that the entire function f we just constructed is the unique extremal for Theorem 2 might not be so well-known, so we include this proof which we learned from B. Ya. Levin (compare [7, 15]).

All critical points of our entire function f are real. Let $x_1 < x_2 < \ldots$ be the sequence of positive critical points of f, Then we have $x_1 > a$, and

$$f(x_k) = 1 + (-1)^{k-1}L, \quad \text{and} \quad f(A) = 1 - L. \tag{6}$$

Let g be another real entire function of exponential type 1 such that

$$\sup |g(x) - 1| \leq L \quad \text{for} \quad x \geq A. \tag{7}$$

We may assume without loss of generality that g is odd (otherwise replace it by $(g(x) - g(-x))/2$ which also satisfies (7)). Equations (6) and (7) imply that the graph of g intersects the graph of f on every interval $[x_k, x_{k+1}]$, $k \geq 1$. More precisely, there is a sequence of zeros y_k of $f - g$ (where multiple zeros are repeated according to their multiplicity), which is interlacent with x_k, that is

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \ldots,$$

and *in addition* to those y_j, $f - g$ has at least one zero in $(0, x_1]$. By the well-known theorem [9, VII, Thm. 1], it follows that the meromorphic function

$$F(z) = \frac{1}{z} \prod_{k \in \mathbf{Z} \setminus \{0\}} \frac{1 - z/x_k}{1 - z/y_k}$$

has imaginary part of constant sign in the upper half-plane, and of opposite sign in the lower half-plane. This implies that

$$F(re^{i\theta}) = O(r), \tag{8}$$

when $r \to +\infty$, uniformly with respect to θ for $\epsilon \leq \theta \leq \pi - \epsilon$, for every $\epsilon > 0$. Similar estimate holds in the lower half-plane.

As $(f - g)(y_k) = 0$, we have

$$\frac{f - g}{f'} = P/F, \tag{9}$$

where P is an entire function of exponential type.

It is easy to see that the left hand side of (9) is bounded for $|\Im z| \geq 1$. Indeed, Phragmén and Lindelöf give $|f(z) - g(z)| \leq C_1 \exp|\Im z|$, while f' has only real zeros and approaches $L\cos(x \pm \alpha))$ as $x \to \pm\infty$, where α is some real constant. It follows that $|f'(z)| \geq C_2 \exp|\Im z|$ for $|\Im z| > 1$, so $(f - g)/f'$ is bounded for $|\Im z| > 1$.

So we conclude from (8) that $P(z) = O(|z|)$ and this contradicts the fact that P has at least two zeros, unless $P = f - g = 0$. This completes the proof.

Proof of Theorem 1. We recall that $B_m = B_m(a)$, $\Omega_m = \Omega_{B_m,m}$ and the conformal maps of the first quadrant Q onto Ω_m were defined before Theorem 4. We are going to prove (3) first, which is the same as

$$B_m = \left(m + \frac{1}{2}\right)\log\frac{1+a}{1-a} + \frac{1}{2}\log m + \frac{1}{2}\log\frac{2a}{1-a^2} + c + o(1), \tag{10}$$

as $m \to \infty$ and a is fixed. Here c is an absolute constant.

We need some auxiliary conformal maps. Let $\psi : Q \to Q$ be defined by

$$\psi(z) = \frac{A_m}{a}\sqrt{\frac{z^2 - a^2}{1 - z^2}},$$

where

$$A_m = \left(m + \frac{1}{2}\right)\log\frac{1+a}{1-a} > 0 \tag{11}$$

The reason for such choice of A_m will be seen later. Then ψ gives the following boundary points correspondence:

$$\psi : (0, a, 1, \infty) \mapsto (iA_m, 0, \infty, iA_m/a).$$

The function $\Phi_m(z) = i\phi_m \circ \psi^{-1}(-iz)$ maps the *second* quadrant onto $i\Omega_m$, and sends the positive imaginary axis onto the interval $\ell = (0, i\pi(m+1))$. By reflection, we extend Φ_m to a map from the upper half-plane onto the region $i(\Omega_m \cup \overline{\Omega_m}) \cup \ell$. This extended map Φ_m gives the following boundary correspondence:

$$\Phi_m : (-C_m, -A_m, 0, A_m, C_m) \mapsto (-\infty, -B_m, 0, B_m, +\infty),$$

where we set $C_m = A_m/a$.

Now we introduce the map $h_m(z) = \Phi_m(z + A_m) - B_m$. Our first goal is to show that the sequence h_m tends to a limit, and to describe this limit. The boundary correspondence under h_m is this:

$$h_m : (-C_m - A_m, -2A_m, -A_m, 0, C_m - A_m) \mapsto (-\infty, -2B_m, -B_m, 0, +\infty).$$

We represent h_m as the Schwarz integral of its imaginary part:

$$h_m(z) = \left(m + \frac{1}{2}\right) \log \frac{1 + za/((1+a)A_m)}{1 - za/((1-a)A_m)} + \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{1}{t}\right) v_m(t)dt, \tag{12}$$

where

$$v_m(t) = \begin{cases} \Im h_m(t), & t \in [-C_m - A_m,\, C_m - A_m], \\ \pi/2, & t \notin [-C_m - A_m,\, C_m - A_m]. \end{cases}$$

Our choice of A_m in (11) implies that the first summand in the right hand side of (12) has a limit

$$\lim_{m\to\infty} \left(m + \frac{1}{2}\right) \log \frac{1 + za/((1+a)A_m)}{1 - za/((1-a)A_m)} = \sigma z,$$

where

$$\sigma = \frac{2a}{(1-a^2)\log((1+a)/(1-a))}. \tag{13}$$

It is easy to see that the integral in (12) converges to a bounded function of the form

$$\frac{1}{\pi} \int \left(\frac{1}{t-z} - \frac{1}{t}\right) \rho_\sigma(t)dt,$$

where ρ_σ is a bounded positive function which we will describe shortly. The image of h_m has a limit Ω^* in the sense of Caratheodory; Ω^* is the region in the upper half-plane above the graph of the function $\arccos e^{-x}$, $x \geq 0$. So

$h_m \to H_\sigma$, where H_σ is the conformal map of the upper half-plane onto Ω^*, $H_\sigma(0) = 0$ and $H_\sigma(z) \sim \sigma z$ as $z \to \infty$.

Thus we have a Schwarz representation

$$H_\sigma(z) = \sigma z + \frac{1}{\pi} \int \left(\frac{1}{t - z} - \frac{1}{t} \right) \rho_\sigma(t) dt,$$

and $\rho_\sigma(t) = \Im H_\sigma(t)$ for real t. Our next goal is to study asymptotics of $B_m = -h_m(-A_m)$. We use the following comparison function

$$g_m(z) = \left(m + \frac{1}{2} \right) \log \frac{1 + za/((1+a)A_m)}{1 - za/((1-a)A_m)} + \frac{1}{2} \int_0^\infty \left(\frac{1}{t - z} - \frac{1}{t} \right) \chi_m(t) dt,$$

where χ_m is the characteristic function of the set $(-\infty, -2(A_m+1)] \cup [2, +\infty)$. We have

$$\lim_{m \to \infty} (h_m(-A_m) - g_m(-A_m)) = -\frac{1}{\pi} \int_0^\infty \left(\rho_\sigma(t) - \frac{\pi}{2} \chi_{[2,\infty)}(t) \right) \frac{dt}{t},$$

and using (11),

$$g_m(-A_m) = -A_m - \frac{1}{2} \log(A_m + 1).$$

Combining these two equations we obtain that

$$B_m = -h_m(-A_m) = A_m + \frac{1}{2} \log A_m + \frac{1}{\pi} \int_0^\infty \left(\rho_\sigma(t) - \frac{\pi}{2} \chi_{[2,\infty)}(t) \right) \frac{dt}{t} + o(1).$$

Using the evident transformation law $H_\sigma(\lambda z) = H_{\sigma\lambda}(z)$ we obtain $\rho_\sigma(\lambda t) = \rho_{\sigma\lambda}(t)$, and therefore,

$$B_m = A_m + \frac{1}{2} \log A_m + \frac{1}{2} \log \sigma + \frac{1}{\pi} \int_0^\infty \left(\rho_1(t) - \frac{\pi}{2} \chi_{[2,\infty)}(t) \right) \frac{dt}{t} + o(1).$$

Substituting the values of A_m and σ from (11) and (13) we obtain (10) with

$$c = \frac{1}{\pi} \int_0^\infty \left(\rho_1(t) - \frac{\pi}{2} \chi_{[2,\infty)}(t) \right) \frac{dt}{t}. \tag{14}$$

The numerical value

$$c = (1/2) \log(2\pi)$$

is obtained from comparison with Bernstein's result (2).

Indeed, in view of (1), (5) and (10) we have

$$\begin{aligned} L_m(a) &= \inf_{\deg q = m} \sup_{[a^2,1]} \sqrt{x}|q(x) - 1/\sqrt{x}| \sim 2e^{-B_m} \\ &= 2(e^{-c} + o(1)) \frac{1-a}{\sqrt{2a}} \left(\frac{1-a}{1+a}\right)^m \frac{1}{\sqrt{m}}. \end{aligned}$$

Comparing this expression for $L_m(a)$ with the expression (2) when $a \to 1$, we obtain (10) with $c = (1/2)\log(2\pi)$.

Proof of Theorem 2. We have to prove that $B = A + (1/2)\log A + c + o(1)$ as $A \to \infty$, where $c = (1/2)\log(2\pi)$.

Let $f_1(z) = \sqrt{z^2 - A^2}$ be the conformal map of the first quadrant Q onto itself, sending A to 0, and $f_1(z) \sim z$ as $z \to \infty$. Then $f_1(0) = iA$. Let f_2 be the conformal map of Q onto Ω_B, $f_2(0) = 0$ and $f_2(z) \sim z$, as $z \to \infty$. We extend f_2 by symmetry, reflecting both domains in the positive ray. So from now on f_2 is defined in the right half-plane. The condition that $f_2(iA) = iB$ defines the number B uniquely. It is easy to see that

$$B \sim A$$

as $A \to \infty$.

Now put $h(z) = if_2(-iz)$ for convenience. This h maps the upper half-plane onto a subregion of the upper halfplane. The boundary of this subregion is asymptotic to the line $\{\Im z = \pi/2\}$ as $|\Re z| \to \infty$.

We have $B = h(A)$, and we wish to find asymptotics of $h(A)$ as $A \to \infty$.

To do this, we use the following comparison function

$$g(z) = z + \int_{A+2}^{\infty} \frac{z}{t^2 - z^2} dt.$$

The integral in the right hand side is the Schwarz formula for an analytic function in the upper half-plane whose imaginary part equals

$$(\pi/2)\chi_{(-\infty,-A-2]\cup[A+2,\infty)}.$$

Our function h has a similar representation in terms of its imaginary part on the real line. Subtracting these two representations, we obtain

$$g(A) - h(A) = \frac{2A}{\pi} \int_0^{\infty} \frac{v(t+A)}{2At + t^2} dt,$$

where $v(t)=\Im(g(t)-h(t))$. Now we claim that $v(t+A)$ tends to a limit v_0, as $A\to+\infty$. This limit is $(\pi/2)\chi_{[2,\infty)}(t)-\Im H(t)$, where and $H(z)=\lim_{A\to+\infty}(h(z+A)-B(A))$ is the conformal map of the upper half-plane onto the region in the upper half-plane above the graph $y=\arccos e^{-x}$, $x\geq 0$ and $y=0$, $x\leq 0$. Notice that $H=H_1$, where H_1 was defined in the proof of Theorem 1.

Thus $g(A)-h(A)\to$ const where the constant is given by

$$\frac{1}{\pi}\int_0^\infty\frac{v_0(t)}{t}dt=\frac{1}{\pi}\int_0^\infty(\frac{\pi}{2}\chi_{[2,\infty)}(t)-\Im H(t))\frac{dt}{t}. \tag{15}$$

The integral is convergent because $v_0(t)=O(\sqrt{t}),\ t\to 0$ and $v_0(t)=O(e^{-t}),\ t\to\infty$. It remains to find the asymptotic behavior of $g(A)$ as $A\to\infty$. We have

$$g(A)=A+\int_2^\infty\frac{A}{2tA+t^2}dt=A+\frac{1}{2}\log(A+1).$$

Combining these results we obtain

$$\begin{aligned}B=h(A) &= A+\frac{1}{2}\log A+\frac{1}{\pi}\int_0^\infty\left(\Im H(t)-\frac{\pi}{2}\chi_{[2,\infty)}(t)\right)\frac{dt}{t}+o(1)\\ &= A+\frac{1}{2}\log A+c+o(1),\end{aligned}$$

where c is the constant from (14). This proves Theorem 2.

Proof of Theorem 5. We choose

$$B=\log m-\log\log m+\log 4, \tag{16}$$

so that $L\sim(\log m)/(2m)$ in view of (5). Consider the conformal map of $\Omega_{B,m}$ onto the half-strip $\Pi=\{z:\Re z\in(0,(m+1)\pi)\}$ by a function f_1 such that $f_1(0)=0$, $f_1((m+1)\pi)=(m+1)\pi$ and $f_1(\infty)=\infty$. Let $B'=f_1(B)$. The function $f_1(z/B)$ tends to the identity as $m\to\infty$, so

$$B'\sim B,\quad B\to\infty.$$

Let f_2 be the (elementary) conformal map of Π onto the first quadrant normalized by $f_2(B')=0$, $f_2(\pi(m+1))=1$ and $f_2(\infty)=\infty$. Then

$$a:=f_2(0)=\frac{1-\exp(B'/(m+1))}{1+\exp(B'/(m+1))}=B'/2(m+1)\sim B/2m\sim(\log m)/(2m).$$

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