Uniqueness of solution of the Dirichlet problem

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I recall what is the Dirichlet problem. Given a region D and a function ϕ defined on ∂D, to find a harmonic function u in D such that

$$\lim_{z \to \zeta} u(z) = \phi(\zeta), \quad \text{for all} \quad \zeta \in \partial D.$$

Suppose that D is bounded, the function ϕ is continuous, and u is the solution of the Dirichlet problem. Then the function

$$\tilde{u}(z) = \left\{ \begin{array}{ll} u(z), & z \in D, \\ \phi(z), & z \in \partial D \end{array} \right.$$

is continuous in the closure $\overline{D} = D \cup \partial D$ of D.

In this case, the Maximum principle for harmonic functions implies that a solution is unique (if it exists).

The condition of continuity of ϕ is too restrictive, see for example the text on Poisson formula. It is desirable to extend the uniqueness at least to piecewise-continuous functions.

Consider the following example:

$$u(z) = \text{Re}\, \frac{1+z}{1-z}, \quad |z| < 1.$$

Exercise: show that

$$\lim_{z \to \zeta} u(\zeta) = 0$$

for all ζ on the unit circle except $\zeta = 1$, but at the point 1 the limit does not exist.

This example shows that some additional conditions are needed for uniqueness in the case that the boundary function is discontinuous. An appropriate condition is *boundedness* of the harmonic function. We have the following generalization of the Maximum Principle:

Phragmén–Lindelöf Principle. *Let D be a bounded domain and u a bounded harmonic function in D. Suppose that*

$$\lim_{z \to \zeta} u(z) = 0, \quad \textit{for all} \quad \zeta \in \partial D \backslash E,$$

where E is a finite set. Then $u = 0$ in D.

Proof. Let $E = \{a_1, \dots, a_n\}$. For every $\epsilon > 0$, consider the function

$$w_\epsilon(z) = u(z) + \epsilon \sum_{j=1}^{n} \operatorname{Log} |z - a_j|.$$

This function is harmonic in D, and

$$\lim_{z \to \zeta} w_\epsilon(z) \leq \epsilon n \operatorname{Log} d, \quad \text{for all} \quad \zeta \in \partial D,$$

where $d = \max_{z,j} |z - a_j|$ (this is finite since D is bounded!). We used that $\operatorname{Log} x$ tends to $-\infty$ as $x \to 0$. So by the Maximum Principle, we have

$$w_\epsilon(z) \leq \epsilon n \operatorname{Log} d, \quad \text{for all} \quad z \in D,$$

and since $\epsilon > 0$ is arbitrary, we conclude that $u \leq 0$ in D. Applying the same to $-u$, we obtain that $u = 0$ in D.

By conformal mapping, this can be extended to unbounded domains.