

Lectures on geometric theory of meromorphic functions, IMPAN, May 2023

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Lecture 1. Introduction. What's the geometric theory of meromorphic functions? Riemann surfaces spread over the sphere. Singularities: some general results. Speiser class and line complexes. Cellular functions and nets. Theorems of Gross and Iversen. Mazurkiewicz metric. General results on singularities. Sullivan's question, and partial results about it. Vinberg's question.

Sources: A. Eremenko, Geometric theory of meromorphic functions, in the book: In the Tradition of Ahlfors-Bers, III, (Contemp. math., 355) AMS, Providence, RI, 2004, pp. 221-230.

R. Nevanlinna, Analytic functions, Springer, 1970.

L. Geyer and S. Merenkov, A hyperbolic surface with a square grid net, J. Anal. Math. 96 (2005), 357–367.

Lecture 1a. Some criteria of hyperbolic type. Theorems of Picard, Ahlfors-Schwarz lemma, theorems of Bloch and Valiron. Zalcman's Lemma and a proof of the Ahlfors Five Islands theorem. The work of Bonk and Eremenko. Geometric characterization of normal and Bloch functions. Functions with bounded spherical derivative.

Sources: R. Robinson, A generalization of Picard's and related theorems, Duke Math. J., 5 (1939) 118–132.

W. Bergweiler, A new proof of the Ahlfors five islands theorem. J. Anal. Math. 76 (1998), 337–347.

M. Bonk and A. Eremenko, Covering properties of meromorphic functions, negative curvature and spherical geometry. Ann. of Math. (2) 152

(2000), no. 2, 551–592. M. Bonk and A. Eremenko, Uniformly hyperbolic surfaces, *Indiana Univ. Math. J.* 49 (2000), no. 1, 61–80.

J. Duval and da Costa, Sur les courbes de Brody dans $P^n(C)$. *Math. Ann.* 355 (2013), no. 4, 1593–1600.

Lecture 2. Parabolic type criteria. Ahlfors, Kobayashi, Nevanlinna–Wittich, Milnor, Doyle, Nevanlinna conjecture, Benjamini-Schramm-Merenkov.

Sources: L. Ahlfors, Sur le type de surface de Riemann, *C. R.*, 201 (1935) 30–32.

H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Springer, 1955.

P. Doyle, On deciding whether a surface is parabolic or hyperbolic. *Geometry of random motion* (Ithaca, N.Y., 1987), 41–48, *Contemp. Math.*, 73, Amer. Math. Soc., Providence, RI, 1988.

P. Doyle, Random walk on the Speiser graph of a Riemann surface. *Bull. Amer. Math. Soc. (N.S.)* 11 (1984), no. 2, 371–377.

I. Benjamini, S. Merenkov, and O. Schramm, A negative answer to Nevanlinna’s type question and a parabolic surface with a lot of negative curvature. *Proc. Amer. Math. Soc.* 132 (2004), no. 3, 641–647.

Lecture 2a. Introduction to Nevanlinna theory. Direct and indirect singularities. Three theorems about them: DCA, Heins, BE. Applications to value distribution (Bergweiler–Eremenko).

Sources: A. Eremenko, Lectures on Nevanlinna theory in the Weizmann Institute.

A. Eremenko, Singularities of inverse functions.

M. Heins, Asymptotic spots of entire and meromorphic functions, *Ann. Math.*, 66 (1957) 430–439.

W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, *Rev. Mat. Iberoamericana* 11 (1995), no. 2, 355–373.

Lecture 3. Class S. Applications to dynamics: non-wandering theorem and structural stability. Class B (Sixsmith). Applications to ODE. Zeros of eigenfunctions (with Shapiro, with Merenkov), Bank–Laine functions. Extremal problems: Markov inequality, 3 critical values, length of lemniscate.

Sources: A. Eremenko and M. Lyubich, Dynamical properties of some classes

of entire functions. *Ann. Inst. Fourier (Grenoble)* 42 (1992), no. 4, 989–1020.

A. Eremenko and A. Gabrielov, Analytic continuation of eigenvalues of a quartic oscillator. *Comm. Math. Phys.* 287 (2009), no. 2, 431–457.

D. Sixsmith, A new characterization of the Eremenko-Lyubich class, *J. Anal. Math.* 123 (2014), 95–105.

Lecture 4. MacLane class. Nets. Applications, Comb functions, Shapiro–Shapiro conjecture, Hayman’s question, Semmler, Bank-Laine functions.

Sources: G. McLane, Riemann surfaces and asymptotic values associated with real entire functions, *Rice Inst. Pamphlet*, Nov. 1952, 93pp.

E. Vinberg, Real entire functions with prescribed critical values, *Problems in group theory and in homological algebra (Russian)*, 127–138, Yaroslavl. Gos. Univ., Yaroslavl, 1989.

A. Eremenko and P. Yuditskii, Comb functions. Recent advances in orthogonal polynomials, special functions, and their applications, 99–118, *Contemp. Math.*, 578, Amer. Math. Soc., Providence, RI, 2012.

A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry. *Ann. of Math.* (2) 155 (2002), no. 1, 105–129.

W. Bergweiler, and A. Eremenko, On the Bank-Laine conjecture, *J. Eur. Math. Soc. (JEMS)* 19 (2017), no. 6, 1899–1909.

W. Bergweiler and A. Eremenko, Quasiconformal surgery and linear differential equations. *J. Anal. Math.* 137 (2019), no. 2, 751–812.

W. Bergweiler, A. Eremenko and L. Rempe, Second order linear differential equations with a basis of solutions having only real zeros.

G. Semmler, Complete interpolating sequences, the discrete Muckenhoupt condition, and conformal mapping. *Ann. Acad. Sci. Fenn. Math.* 35 (2010), no. 1, 23–46.