Classification: Mathematics.

Title:

Oscillation of functions with a spectral gap

Authors and author affiliation:

A. Eremenko (∗) and D. Novikov (∗)

Department of Mathematics (∗)

Purdue University

West Lafayette, IN

47907

USA

Corresponding author:

Alexandre Eremenko,

Department of Mathematics

Purdue University

150 N. University Street

West Lafayette, IN 47907-2067

Character count:

All text characters (in the TEXsource file) 16,858

Equations (9 one-line, one-column $60 \times 9 = 540$)

Total characters 17,358.

Abstract. We prove an old conjecture on oscillation of functions that have a spectral gap at the origin. Suppose that the Fourier transform of a real temperate distribution f on the real line satisfies $\hat{f}(x) = 0$ for $x \in (-a, a)$. Then, when $r \to \infty$, the asymptotic lower density of the sequence of sign changes of f on the intervals $[0, r)$ is at least a/π. This still holds for some wider classes of distributions characterized by their rate of growth at infinity, but if the growth is faster then a certain threshold, the above statement is no longer true.

We say that a real function on the real line has a spectral gap if its Fourier transform is zero in a neighborhood of the origin. In communication engineering such functions are called high-pass signals. It is known that functions with a spectral gap oscillate. The simplest result of this sort goes back to Sturm [1]: *Every real trigonometric polynomial of the form*

$$\sum_{n\geq m}(a_n\cos nt+b_n\sin nt)\not\equiv 0$$

has at least $2m$ *sign changes on every period.* Hurwitz extended this result to arbitrary periodic functions whose first m harmonics vanish. At least three different proofs of the Sturm–Hurwitz theorem can be found in [2] (II-141, III-184 and VI-57), and yet another one in [3].

In this note we discuss generalizations of the Sturm–Hurwitz theorem to non-periodic functions. We use the standard notation for the spaces of Schwartz's distributions, $\mathscr{D}'$ (the dual to the space $\mathscr{D}$ of smooth functions with compact support) and $\mathscr{S}'$ (temperate distributions) on the real line, and we identify $\mathscr{S}'$ with a subspace of $\mathscr{D}'$ as in [4]. A distribution f is called real if (f,ϕ) is real for every real test function ϕ. The *number of sign changes* of a real distribution f on an interval $(0,r)$ is denoted by $s(r,f)$. It is defined as the minimal degree of real polynomials p such that the restriction of pf on $(0,r)$ is a non-negative distribution (measure). For example, the distribution $\delta-1$ has two sign changes on any interval containing zero. It is clear that $s(r,f)$ is an integer-valued increasing function. If $s(r,f)<\infty$ for every r, then the sequence of points of jump of $s(r,f)$ tends to infinity. This sequence (r_k), where each term is repeated according to the magnitude of the jump, will be called the *sequence of sign changes* of f.

The earliest generalization known to us of the Hurwitz theorem to non-periodic case is due to M. G. Krein and B. Levin [5, Appendix II]:

For real measures f of bounded variation on the real line, the condition $\hat{f}(x) = 0,\ x \in (-a, a)$ implies that $S(r, f) \geq 2ar/\pi + O(1),\ r \to \infty$, where

$$S(r, f) = \int_0^r (s(t, f) + s(-t, f))\, dt/t \tag{1}$$

is the "average density function" of sign changes. Recently this result was improved and extended by Ostrovskii and Ulanovskii [6], in particular, they showed that the Krein–Levin theorem remains valid under the weaker assumption that $f(x)/(1+x^2)$ is of bounded variation. In particular, their result applies to all real bounded functions.

In 1965 B. Logan, in his thesis [7] on high-pass signals, conjectured that

$$\liminf_{r \to \infty} s(r, f)/r \geq a/\pi \tag{2}$$

for functions f with a spectral gap $(-a, a)$. The example $f(t) = \cos t$, which has a spectral gap $(-1, 1)$ and $s(r, f) = r/\pi + O(1)$ shows that this estimate is best possible.

Logan proved (2) under the additional assumption that f is a bounded function with *bounded spectrum.* The latter condition, meaning that $\operatorname{supp} \hat{f}$ is bounded, is crucial for his proof to work. Logan's conjecture was repeated in several places, including [8, 9]. In the commentary to this conjecture in [8],

B. Kuksin sketched a proof of (2) for almost all trajectories f of a Gaussian stationary process whose correlation function has a spectral gap $(-a, a)$ (this implies that almost all trajectories have this spectral gap as well).

Our main result confirms Logan's conjecture for a wide class of distributions which contains $\mathscr{S}'$. A smooth positive even function on the real line, concave[1] on some ray $[t_0, +\infty)$, will be called a *weight.*

THEOREM 1 *Let ω be a weight satisfying*

$$\int_{-\infty}^{\infty} \frac{\omega(t)}{1+t^2} dt < \infty, \tag{3}$$

and $f \in \mathscr{D}' \backslash \{0\}$ a real distribution with the property

$$e^{-\omega} f \in \mathscr{S}'. \tag{4}$$

If f has a spectral gap $(-a, a)$ then (2) holds.

To discuss the role of the growth conditions (4) and (3), we recall a general definition of spectrum, which goes back to Carleman [11]. Suppose that a distribution $f \in \mathscr{D}'$ satisfies

$$\exp(-\epsilon\sqrt{1+x^2}) f(x) \in \mathscr{S}', \quad \text{for every} \quad \epsilon > 0, \tag{5}$$

which is a weaker condition than (4). Let η be a smooth function with the properties: $\operatorname{supp} \eta \subset [-1, \infty)$ and $\eta(t) + \eta(-t) \equiv 1$. Then the two halves of the

[1]The assumption of concavity is made only to simplify the discussion. Each of our results holds under some weaker assumptions on the regularity of the weight, for example, Theorem 1 holds for those weights that satisfy the conditions of the Beurling–Malliavin Multiplier Theorem [10].

Fourier transform of f

$$F^+(z) = (f(t), \eta(-t)e^{-itz}) \quad \text{and} \quad F^-(z) = -(f(t), \eta(t)e^{-itz})$$

are holomorphic in the upper and lower half-planes of the complex z-plane, respectively. The pair of functions $F = (F^+, F^-)$ is called the Fourier–Carleman transform of f. If $f \in \mathscr{S}'$ then the functions $F^\pm$ have boundary values in the sense $\mathscr{S}'$, and $\hat{f}$ is the difference of these boundary values. Moreover, if the restriction of $\hat{f}$ is zero on some interval I on the real line, then F^+ and F^- are analytic continuations of each other across this interval I. This ensures that the following definition of the spectrum is consistent with the usual one as the support of the Fourier transform.

DEFINITION 1 *The spectrum of a distribution f that satisfies* (5) *is the complement of the maximal open set $U \subset \mathbf{R} \cup \{\infty\}$ such that $F^\pm$ are analytic continuations of each other through U. A spectral gap is an interval $(-a, a) \subset U$.*

Changing η in the definition of $F^\pm$ adds an entire function to both F^+ and F^-, so the spectrum depends only on f. A theorem of Pólya's characterizes the class of distributions satisfying (5) with spectrum on an interval $[-b, b]$: this class coincides with the class of entire functions of exponential type whose indicator diagrams are contained in the interval $[-ib, ib]$ of the imaginary axis [5]. Definition 1 relates our subject with the classical topic of analytic continuation [12].

It turns out that the growth conditions (4) and (3) are crucial for validity of Theorem 1, and in certain sense these conditions are best possible.

Theorem 2 *Let ω be a weight with divergent integral* (3). *Given two positive numbers $a < b$, there exists a real function $f \neq 0$ with the property $|f| < \exp \omega$, such that the spectrum of f is contained in $[-b, -a] \cup [a, b]$, but* (2) *fails.*

Convergence or divergence of the integral (3) is a fundamental dichotomy in Harmonic Analysis, [13, 14, 5, 15, 16].

Thus (2) may fail for distributions that satisfy only (5). However, developing the ingenious idea of Levin, one can estimate from below the *average density function* (1) of sign changes of such distributions.

Theorem 3 *If a real distribution $f \neq 0$ satisfies* (5), *and has a spectral gap $(-a, a)$, then*

$$\liminf_{r \to \infty} S(r, f)/r \geq 2a/\pi, \tag{6}$$

where $S(r, f)$ is defined in (1).

We notice that the spectrum of a function f satisfying (5) may consist of the single point ∞. This happens when $F^{\pm}$ are restrictions of an entire function F to the upper and lower half-planes. To obtain such example, take any entire function that satisfies $F(z) = O(z^{-2})$ as $z \to \infty$ on the sets $\{z : |\mathrm{Im}\, z| \geq \epsilon\}$, for every $\epsilon > 0$, and define f as the inverse Fourier–Carleman transform:

$$f(t) = -\frac{1}{2\pi} \int_{\gamma} F(z) e^{itz} dz,$$

where γ is the oriented boundary of the strip $\{z : |\mathrm{Im}\, z| = \epsilon\}$, and $\epsilon > 0$. Our Theorem 3 implies that for such functions f, $S(r, f)/r \to \infty$.

Sketch of the proof of Theorem 1. The proof is quite technical, so we only describe the main steps.

First we reduce the problem to the case that f is real analytic, replacing f by the convolution $f_t = K_t * f$ with the heat kernel $K_t = (\pi t)^{-1/2} \exp(-x^2/t)$, $t > 0$. This does not change the spectrum and *does not increase* the number of sign changes. The latter property, discovered by Sturm, was subject of a series of beautiful generalizations of Kellogg [17], Pólya [3], Shoenberg [18], and others. However the existing results are not sufficient for our purposes because they apply only to functions with finitely many changes of sign on the whole real axis, while we need to control this number on every interval $[0, r]$. To prove what is needed, we extend the original approach of Sturm based on the qualitative theory of the heat equation.

The next step is representing f as the real part of a function h holomorphic in the upper half-plane, whose spectrum belongs to $[a, \infty)$. Such representations were also used by Levin and Logan. To obtain them one multiplies the Fourier transform of f by the Heaviside function and takes the inverse transform. After this, it is enough to estimate from below (a suitable continuous branch of) the argument $\arg h(t)$ as $t \to +\infty$. Indeed, assuming for simplicity that $h(t) \neq 0$ for all real t, we notice that every turn of $h(t)$ counterclockwise around zero forces $f = \mathrm{Re}\, h$ to change sign at least twice.

The growth restrictions (4) and (3) on f are used to prove that $h \in N$, the Nevanlinna class of functions of bounded characteristic in the upper half-plane

[19]. To show this we use a *multiplier*, that is an entire function g of arbitrarily small exponential type, with only real zeros, and such that $g \exp \omega$ is bounded on the real line. The existence of multipliers for nice weights was known to Paley and Wiener [13]; the result in this direction with minimal assumptions about the weight is due to Beurling and Malliavin [10, 15, 16]. Choosing a multiplier g, we arrange that gf is integrable on the real line. Then we represent gf as a real part of an analytic function h_1 in the upper half-plane using the Cauchy integral. This function h_1 can be shown to belong to a Hardy space, and thus to the space N. As the multiplier belongs to N as well, and has only real zeros, we conclude that $h = h_1/g \in N$.

Next we use the Nevanlinna representation of functions of the class N:

$$h(z) = e^{ia'z} B(z) e^{u(z)+iv(z)}, \quad \operatorname{Im} z > 0, \tag{7}$$

where a' is a real number, B a Blaschke product in the upper half-plane, u the Poisson integral of $\log |h(x)|$ and v a harmonic conjugate of u. This function v can be expressed by a formula which is called the Hilbert transform of u.

It is the number a' in (7) that is responsible for the growth of $\arg h(x)$ on the real line. The information about the spectrum of h implies that $a' \geq a$, which gives (2), after an estimation of the contribution to $\arg h(x)$ of the rest of the terms in (7). The Blaschke product has increasing argument, which can only help, and the contribution of v is estimated using the Kolmogorov's inequality for the Hilbert transform [15].

We thank A. Gabrielov, I. Ostrovskii and M. Sodin for their remarks. A. E.

was supported by the NSF grant DMS 0100512 and by the Humboldt Foundation, and D. N. by the NSF grant DMS 0100512 and by the Humboldt Foundation.

References

[1] Sturm, Ch., (1836) *J. de Math. pures et appl.* **1**, 373–444.

[2] Pólya, G. and Szegö, G., (1925) *Aufgaben and Lehrsätze aus der Analysis*, B I–II (Springer, Berlin).

[3] Pólya, G., (1933) *Z. angew. Math. Mech.* **13**, 125–128.

[4] Hörmander, L., (1983) *The analysis of linear partial differential operators*, vol. I and II (Springer, Berlin).

[5] Levin, B., (1980) *Distribution of zeros of entire functions*, (AMS, Providence RI, Revised edition).

[6] Ostrovskii, I. and Ulanovskii, A., (2002) to appear in *C. R. Acad. Sci., Paris.*

[7] Logan, B., (1965) *Properties of high-pass signals*, (Thesis, Dept. Electrical Engineering, Columbia Univ.).

[8] Задачи Арнольда, (2000) Фазис, Москва (Russian, English translation in preparation).

[9] Shapiro, H., (2001) in *Twentieth Century Harmonic Analysis*, ed. Byrnes, J., (Kluwer, Dordrecht), 383–384.

[10] Beurling, A., and Malliavin, P., (1967) *Acta math.* **118**, 291–309.

[11] Carleman, T., (1944) *L'Intégrale de Fourier et questions qui s'y rattachent*, (Inst. Mittag-Leffler).

[12] Bieberbach, L., (1955) *Analytische Fortsetzung*, (Springer, Berlin).

[13] Paley, R. and Wiener, N., (1934) *Fourier transform in the complex domain* (AMS, Providence, RI).

[14] Beurling, A., (1939) in *9 Congr. Math. scand.,* (Helsingfors), 345–366.

[15] Koosis, P., (1988) $\int_{-\infty}^{\infty} \frac{\log M(t)}{1+t^2} dt$. *The Logarithmic Integral*, 2 vol. (Cambridge UP).

[16] Havin, V., and Joricke, B., (1994) *The Uncertainty Principle in Harmonic Analysis* (Springer, NY).

[17] Kellogg, O., (1916) *Amer. J. Math.* **38**, 1–5.

[18] Schoenberg, I., (1948) *Proc. Nat. Acad. Sci. USA* **34**, 164–169.

[19] Zygmund, A., (1959) *Trigonometric series*, vol. I (Cambridge UP).