Poisson's formula

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A problem of fundamental importance to mathematical physics is the

Dirichlet's Problem. Let D be a region, and ϕ a function defined on the boundary ∂D. Find a harmonic function $u(z)$ in D with prescribed boundary values,

$$\lim_{z\to\zeta} u(z) = \phi(\zeta), \quad \zeta \in \partial D. \tag{1}$$

For example, let D be a plate, and we know the temperature on the boundary. Find the temperature at every point inside the plate. Or suppose that we know the electrostatic potential on the boundary, and want to find the potential inside.

In this text, I will explain how to solve this problem in the upper half-plane

$$H = \{z : \operatorname{Im} z > 0\}.$$

The boundary is the real line: $\partial H = \mathbf{R}$.

Consider the function $\operatorname{Arg} z$ in H. We know that this function is harmonic (it is the imaginary part of $\operatorname{Log} z$). The boundary values are 0 on the positive ray and π on the negative ray.

Now for any $a < b$ on the real line, consider the function

$$\phi_{a,b}(\zeta) = \begin{cases} 0, & \zeta > b, \\ 1, & a < \zeta < b, \\ 0, & \zeta < a. \end{cases}$$

It is easy to see that the following harmonic function solves the Dirichlet problem with boundary values ϕ:

$$u_{a,b}(z) = \frac{1}{\pi}\left(\operatorname{Arg}(z-b) - \operatorname{Arg}(z-a)\right).$$

Here we used the crucial fact that a linear combination of harmonic functions is harmonic. This is because the Laplace equation is *linear*.

We should notice that equation (1) does not hold at two points: a and b, which should not be surprising since our boundary function is not continuous.

Now consider a piecewise-constant boundary function: let

$$a_0 < a_1 < \ldots < a_n,$$

and

$$\phi(\zeta) = \begin{cases} b_j, & a_{j-1} < \zeta < a_j,\ 1 \leq j \leq n \\ 0, & \zeta > a_n \quad \text{or} \quad \zeta < a_0. \end{cases} \tag{2}$$

This is a linear combination of functions of the form (2), namely

$$\phi(\zeta) = \sum_{j=1}^{n} b_j \phi_{a_{j-1},a_j}(\zeta).$$

So the solution of the Dirichlet problem with boundary values ϕ as in (2) is

$$u(z) = \sum_{j=1}^{n} b_j u_{a_{j-1},a_j}(\zeta). \tag{3}$$

This suggests how to solve the problem in general: every continuous function can be approximated by step functions of the form (2). To obtain a nice formula, we first express $u_{a,b}$ in terms of x and y (where $z = x + iy$):

$$u_{a,b}(x+iy) = \frac{1}{\pi}\left(\arctan\frac{y}{x-b} - \arctan\frac{y}{x-a}\right).$$

Since

$$\frac{d}{dt}\arctan\frac{y}{x-t} = \frac{y}{(x-t)^2+y^2},$$

we can use the Newton-Leibniz formula and rewrite the expression for $u_{a,b}$ as

$$u_{a,b}(x+iy) = \frac{y}{\pi}\int_a^b \frac{dt}{(x-t)^2+y^2}.$$

Combining this with the formula (3), we obtain the final result

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\phi(t)dt}{(x-t)^2+y^2},$$

which is called the Poisson formula for solution of the Dirichlet problem in the upper half-plane.

We did not address the accurate justification of convergence under the integral sign. The formula really holds for continuous functions f such that the integral

$$\int_{-\infty}^{\infty} \frac{|f(t)|dt}{1+t^2}$$

is convergent.

Also, solution of the Dirichlet problem is not unique, unless we impose extra conditions. For example, the function $u(x,y)=y$ has zero boundary values, so it can be added to any solution. One extra condition which implies uniqueness is that $u(x,y)$ is *bounded* in the upper half-plane.