New counterexamples to pole placement by static output feedback

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Abstract

We consider linear systems with inner state of dimension n, with m inputs and p outputs, such that $n = mp$, $\min\{m, p\} = 2$ and $\max\{m, p\}$ is even. We show that for each (m, n, p) satisfying these conditions, there is a non-empty open subset U of such systems, where the real pole placement map is not surjective. It follows that for systems in U, there exist open sets of pole configurations which cannot be assigned by any real output feedback.

1. Introduction

Let $\mathbf{F}$ be one of the fields $\mathbf{C}$ (complex numbers) or $\mathbf{R}$ (real numbers). For fixed positive integers m, n, p, we consider matrices A, B, C and K with entries in $\mathbf{F}$, of sizes $n \times n$, $n \times m$, $p \times n$ and $m \times p$, respectively. The space Poly_n of monic polynomials of degree n with coefficients in $\mathbf{F}$ is identified with $\mathbf{F}^n$, using coefficients as coordinates. For fixed A, B and C with entries in $\mathbf{F}$, the *pole placement map* $\chi = \chi_{A,B,C}$ is defined as

$$\chi : \mathrm{Mat}_{\mathbf{F}}(m \times p) \to \mathrm{Poly}_n \simeq \mathbf{F}^n, \quad \chi(K) = \phi_K,$$

where

$$\phi_K(s) = \det(sI - A - BKC). \tag{1}$$

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It will be called real or complex pole placement map, depending on $\mathbf{F}$.

Our primary interest is in the real pole placement map. We briefly explain how it arises in control theory. (A general reference for this theory is [2], and a survey of the pole placement problem is [4]). A linear system with static output feedback is described by a system of equations

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\\ u &= Ky.\end{aligned}$$

Here the state x, the input u and the output y are functions of a real variable t (time), with values in $\mathbf{R}^n$, $\mathbf{R}^m$, and $\mathbf{R}^p$, respectively, the dot denotes the derivative with respect to t, and A, B, C, K are real matrices of sizes specified above. The matrix K is usually called a *gain* matrix, or a compensator. Eliminating u and y gives

$$\dot{x} = (A + BKC)x,$$

thus ϕ_K is the characteristic polynomial of the system. The pole placement problem is:

Given real A, B, C and a set of points $\{s_1, \ldots, s_n\} \subset \mathbf{C}$, symmetric with respect to the real axis, to find real K, so that the zeros of ϕ_K are exactly $s_1, \ldots, s_n$.

This amounts to finding real solutions of n algebraic equations with mp unknowns, the entries of K. For any dimensions m, n, p, there are systems (A, B, C), such that the map $\chi_{A,B,C}$ is not surjective (for example, if $B = 0$). So we consider generic systems (A, B, C). We say that for given m, n, p, the (real or complex) pole placement map is *generically surjective* if there exists an open dense set V in the space $\mathrm{Mat}_{\mathbf{F}}(n \times n) \times \mathrm{Mat}_{\mathbf{F}}(n \times m) \times \mathrm{Mat}_{\mathbf{F}}(p \times n)$, such that for $(A, B, C) \in V$, the map $\chi_{A,B,C}$ is surjective.

All topological terms in this paper refer to the usual topology.

We recall the known results on generic surjectivity. The case $\min\{m, p\} = 1$ is completely understood (see, for example, [4, 8]). From consideration of dimensions follows that $mp \geq n$ is a necessary condition of generic surjectivity, complex or real. If $n > mp$, then the real pole placement map is generically surjective [12, 7]. So from now on we restrict our attention to the case $n = mp$.

Theorem A [3] *For $n = mp$, the complex pole placement map is generically surjective. Moreover, it is a finite map of degree*

$$d(m,p) = \frac{1!2!\dots(p-1)!\,(mp)!}{m!(m+1)!\dots(m+p-1)!}.$$

The numbers $d(m,p)$ occur as the solution of the following problem of enumerative geometry: how many m-subspaces intersect mp given p-subspaces in $\mathbf{C}^{m+p}$ in general position? The answer $d(m,p)$ was obtained by Schubert in 1886 (see, for example, [6]).

One can deduce from Theorem A that the real pole placement map is generically surjective if $d(m,p)$ is odd. This number is odd if and only if one of the following conditions is satisfied [1]: a) $\min\{m,p\} = 1$, or b) $\min\{m,p\} = 2$ and $\max\{m,p\} + 1$ is an integral power of 2.

If $n = mp$, the real pole placement map was known not to be generically surjective in the following two cases: a) $m = p = 2$ [13], and b) $\min\{m,p\} = 2$, $\max\{m,p\} = 4$. For the case b), a rigorous computer assisted proof was given in [8]. It disproved an earlier conjecture, that a) is the only case when $n = mp$ and the real pole placement map is not generically surjective.

In this paper we show that infinitely many more cases exist.

Theorem 1 *If $n = mp$, $\min\{m,p\} = 2$, and $\max\{m,p\}$ is even, then the real pole placement map is not generically surjective.*

Our proof of Theorem 1 gives explicitly a system (A,B,C), and a point $w \in \mathrm{Poly}_n$, such that for any (A',B',C') close to (A,B,C), the real pole placement map $\chi_{A,B,C}$ omits a neighborhood of w.

In Section 2, we derive Theorem 1 from a fact about rational functions, Proposition 1, which is proved in Section 3. The proof uses a recent result on rational functions from [5], and we outline its proof in Section 4.

2. A class of linear systems

We begin with a well-known transformation of the expression (1). We factorize the rational matrix-function $C(sI-A)^{-1}B$, as

$$C(sI-A)^{-1}B = D(s)^{-1}N(s), \quad \det D(s) = \det(sI-A), \qquad (2)$$

where D and N are polynomial matrix-functions of sizes $p \times p$ and $p \times m$, respectively. For the possibility of such factorization we refer to [2, Assertion

22.6]. Using (2), and the identity $\det(I - PQ) = \det(I - QP)$, which is true for all rectangular matrices of appropriate dimensions, we write

$$\begin{aligned}\phi_K(s) &= \det(sI - A - BKC) = \det(sI - A)\det(I - (sI - A)^{-1}BKC)\\ &= \det(sI - A)\det(I - C(sI - A)^{-1}BK)\\ &= \det D(s)\det(I - D(s)^{-1}N(s)K) = \det(D(s) - N(s)K).\end{aligned}$$

This can be rewritten as

$$\phi_K(s) = \det\left([D(s), N(s)]\left[\begin{array}{c} I \\ -K \end{array}\right]\right). \tag{3}$$

To prove Theorem 1, we set $p = 2$. This does not restrict generality in view of the symmetry of our problem with respect to the interchange of m and p, see, for example, [10, Theorem 3.3]. We consider a system (A, B, C) represented by

$$[D(s), N(s)] = \left[\begin{array}{ccccc} s^{m+1} & s^m & \dots & s & 1 \\ (m+1)s^m & ms^{m-1} & \dots & 1 & 0 \end{array}\right]. \tag{4}$$

The matrices A, B, C can be recovered from $[D, N]$, by [2, Theorem 22.18]. Let $K = (k_{i,j})$. Introducing polynomials

$$\begin{aligned} f_{1,K}(s) &= s^{m+1} - k_{1,1}s^{m-1} - \ldots - k_{m,1} \qquad \text{and} \\ f_{2,K}(s) &= s^m - k_{1,2}s^{m-1} - \ldots - k_{m,2}, \end{aligned}$$

we can write (3) as

$$\phi_K = f_{1,K}f'_{2,K} - f'_{1,K}f_{2,K} = W(f_{1,K}, f_{2,K}). \tag{5}$$

If we consider the rational function $f_K = f_{2,K}/f_{1,K}$, and suppose the Wronskian determinant $W(f_{1,K}, f_{2,K})$ has no multiple zeros, then $\deg f_K = m+1$, and zeros of $W(f_{1,K}, f_{2,K})$ coincide with the critical points of f_K. The map $K \mapsto f_K$ sends matrices from $\mathrm{Mat}_{\mathbf{C}}(m \times 2)$ to rational functions of degree at most $m + 1$, satisfying

$$f_K(z) = z + O(z^{-1}), \quad z \to \infty. \tag{6}$$

To real matrices correspond real functions.

To prove Theorem 1, we first refer to the following general result [8, Theorem 3.1]:

Suppose that $n = mp$. If there exists a real system (A, B, C) and a real polynomial $\phi \in \mathrm{Poly}_n$ such that $\chi^{-1}_{A,B,C}(\phi)$ consists of $d(m, p)$ distinct complex points, none of them real, then the real pole placement map is not generically surjective.

So, to prove Theorem 1, it is enough to find a system (A, B, C), and a point $w \in \mathrm{Poly}_{2m}$ such that the full preimage $\chi^{-1}(w)$ of this point under the complex pole placement map $\chi = \chi_{A,B,C}$ consists of the maximal possible number $d(m, 2)$ points, and none of these points is real.

Thus Theorem 1 is a corollary from the following

Proposition 1 *Let m be even, S a circle symmetric with respect to the real line, $\{s_1, \ldots, s_{2m}\}$ a subset of S, symmetric with respect to the real line, and $s_{2m} \in \mathbf{R}$. Then there exist exactly $d(m, 2)$ rational functions of degree $m+1$, having $\{s_1, \ldots, s_{2m}\}$ as their critical set, satisfying (6), and none of these functions is real.*

3. Proof of Proposition 1

Two rational functions f and g are called equivalent if $f = \ell \circ g$, where ℓ is a fractional-linear transformation. Equivalent rational functions have the same critical points. Two rational functions normalized as in (6) are equivalent if and only if they are equal. Proposition 1 will be deduced from the following result [5].

Proposition 2 *Given $2m$ points on a circle S in the Riemann sphere, there are $d(m, 2)$ pairwise non-equivalent rational functions of degree $m+1$, having these critical points, and mapping S into itself.*

Proof of Proposition 1. By Proposition 2, there are $d(m, 2)$ normalized rational functions of degree $m + 1$, having the critical set $\{s_1, \ldots, s_{2m}\}$, and mapping S into some circles. It is clear that for each of these functions, all critical points are simple. It remains to prove that none of these functions is real. Suppose the contrary, and let f be a real rational function of degree $m+1$ having $\{s_1, \ldots, s_{2m}\}$ as its critical set, and $f(S)$ is a circle. Then $f(S)$ is symmetric with respect to $\mathbf{R}$, and thus $\gamma = f^{-1}(f(S))$ is also symmetric with respect to $\mathbf{R}$. The set γ is a one dimensional real analytic variety in the plane $\mathbf{C}$, and it is smooth everywhere, except at the points $s_1, \ldots, s_{2m}$ on the circle S. Removing these points $s_1, \ldots, s_{2m}$ from γ leaves disjoint open analytic arcs, whose closures are called the *edges* of γ. Exactly 4 edges meet at every s_j, two of them are arcs of S, and the other two are not. Let $\overline{D}$ be the closed disc bounded by S. Then $\gamma \cap \overline{D}$ consists of S and of m edges whose interiors belong to the open disc D, bounded by S. These m edges are called *chords*. Chords are disjoint, and each of them contains two points of the set $s_1, \ldots, s_{2m}$ as its endpoints.

Consider the chord γ_0 with one endpoint at $s_{2m} \in S \cap \mathbf{R}$. This chord has to be symmetric with respect to $\mathbf{R}$, because no other chord can contain s_{2m}. It follows that $\gamma_0 = \overline{D} \cap \mathbf{R}$. Then the intersection $\overline{D}^+$ of $\overline{D}$ with the open upper half-plane contains $m-1$ of the points s_j, which is an odd number. But this is a contradiction, because these $m-1$ points are connected pairwise by disjoint chords in $\overline{D}^+$. This proves Proposition 1.

4. Rational functions whose critical points belong to a circle

If one replaces the circle S by the real line, (which is a circle in the Riemann sphere), Proposition 2, and Theorem A with $p = 2$ imply: *Every rational function whose critical points are real is equivalent to a real rational function.* This means something opposite to Theorem 1, namely: the pole placement problem for a system (4), with *real* points $\{s_1, \ldots, s_{2m}\}$ always has $d(m, 2)$ real solutions. This is a special case of the B. and M. Shapiro conjecture, proved in [5]. A comprehensive discussion of this conjecture, including numerical evidence, is contained in [10]. We also mention that F. Sottile [9] used systems similar to (4) with arbitrary (m, p) to show that, for any (m, p) and $n = mp$, all $d(m, p)$ solutions of a pole placement problem can be real.

Now we explain the ideas behind the proof of Proposition 2. Without loss of generality, we may assume that $S = \mathbf{T} = \{s : |s| = 1\}$, the unit circle. The word "symmetry" in what follows always refers to the symmetry with respect to $\mathbf{T}$. Let R be the class of all rational functions f of degree $m+1 \geq 3$, such that $f(\mathbf{T}) \subset \mathbf{T}$, $f(1) = 1$, $f'(1) = 0$, all critical points of f are simple and belong to $\mathbf{T}$. We first describe a convenient parametrization of a subclass of properly normalized functions in R. As in the proof of Proposition 1 in Section 3, we consider a *net*, $\gamma(f) = f^{-1}(\mathbf{T})$. It coincides with the 1-skeleton of a cell decomposition of $\overline{\mathbf{C}}$, whose 2-dimensional cells are components of $f^{-1}(\overline{\mathbf{C}} \backslash \mathbf{T})$, and 0-dimensional cells (vertices) are the critical points of f. It is easy to see that this cell decomposition has the following properties:

N1 γ is symmetric with respect to $\mathbf{T}$ and contains $\mathbf{T}$,

N2 all $2m$ vertices belong to $\mathbf{T}$ and have order 4,

N3 the point $1 \in \mathbf{T}$ is a vertex.

Such cellular decompositions are uniquely determined by their 1-skeletons. Two nets γ_1 and γ_2 are called *equivalent* if there exists an orientation-preserving symmetric homeomorphism $h : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, fixing the point 1, and such that $h(\gamma_1) = \gamma_2$. We *label* each edge e of our net $\gamma(f)$ by a positive number $p(e)$, the length of the arc $f(e) \subset \mathbf{T}$. Then

$$\sum_{e \in \partial G} p(e) = 2\pi \quad \text{for every 2-dimensional cell } G. \tag{7}$$

A *labeled net* is a 1-skeleton of a cell decomposition of $\overline{\mathbf{C}}$ satisfying N1-N3, equipped with a non-negative symmetric function p on the set of edges, satisfying (7). If p is strictly positive, we call the labeling non-degenerate. Using the Uniformization Theorem (for the sphere) we prove: *To each labeled net corresponds a rational function $f_{\gamma,p}$, such that for non-degenerate p we have $\gamma \sim f_{\gamma,p}^{-1}(\mathbf{T})$, and $p(e) = \mathrm{length} f(e)$, for every edge e of γ.*
This result gives a parametrization of a properly normalized subclass of R by pairs $([\gamma], p)$, where $[\gamma]$ is a class of equivalent nets, and p a non-degenerate labeling. This parametrization has an advantage that it separates topological information about f, described by the class $[\gamma]$, from continuous parameters p, in such a way that the space of continuous parameters is topologically trivial. Indeed, the space of all non-degenerate labelings of a fixed net is a convex polytope L_γ, described by equations (7), the symmetry conditions, and the condition $p > 0$. The set of all labelings of γ is the closure $\overline{L}_\gamma$ of L_γ. For a fixed net γ and every $p \in L_\gamma$ we take the critical set of $f_{\gamma,p}$, which, in the case of non-degenerate p, is a sequence of $2m$ distinct points on the unit circle, containing the point 1. We denote the space of all such sequences by Σ_γ. It can be identified with an open convex polytope of the same dimension as L_γ. Thus for every γ we have a map $\Phi_\gamma : L_\gamma \to \Sigma_\gamma$. We want to show that it is surjective, using a "continuity method," going back to Poincaré and Koebe. First we extend Φ_γ to a continuous map of the closed polytopes

$$\Phi_\gamma : \overline{L}_\gamma \to \overline{\Sigma}_\gamma.$$

The proof of continuity of the extended map uses classical function theory: Picard's and Montel's theorems. The surjectivity of Φ_γ follows from careful analysis of its boundary behavior. The purpose of this analysis is to show that full preimages of the closed faces of $\overline{\Sigma}_\gamma$ are topologically trivial, that is they have the same homology groups as a one-point space. In particular these preimages are non-empty and connected. As the preimages of the closed faces can be rather complicated, this is done in several steps. First we describe the preimages of *open faces*, which are much simpler because they are convex. Then we consider chains $A_1, A_2, \ldots, A_k$ of open faces of $\overline{\Sigma}$, such that $\overline{A}_1 \supset \overline{A}_2 \supset \ldots \supset \overline{A}_k \neq \emptyset$, and show that, for each such chain, the preimages of closures of faces have non-empty intersection:

$$\overline{\Phi_\gamma^{-1}(A_1)} \cap \ldots \cap \overline{\Phi_\gamma^{-1}(A_k)} \neq \emptyset.$$

This is enough to deduce that the preimages of closed faces are topologically trivial. Once this is established, we use the following fact.

Lemma 1 *Let $\Phi : \overline{L} \to \overline{\Sigma}$ be a continuous map of closed convex polytopes of the same dimension. If the preimage of every closed face of $\overline{\Sigma}$ is (non-empty and) topologically trivial, then Φ is surjective.*

Thus for every class of nets γ and every critical set in Σ_γ we have a rational function $f \in R$, having this critical set, and $\gamma \sim f^{-1}(\mathbf{T})$. As for equivalent functions f_1 and f_2 we have $\gamma(f_1) = \gamma(f_2)$, there are at least as many equivalence classes of functions in R, sharing a given critical set, as the number of classes of nets. So the proof of Proposition 4 is completed by simple combinatorics:

Lemma 2 *The number of classes of nets on $2m$ vertices is $d(m, 2)$, the d-th Catalan number.*

For this we refer to [11, Exercise 6.19 n].

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