New counterexamples to pole placement by static output feedback

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Abstract

We consider linear systems with m inputs, p outputs and state of dimension n, such that $n = mp$, $\min\{m,p\} = 2$ and $\max\{m,p\}$ is even. We show that for each (m,n,p) satisfying these conditions, there is a non-empty open subset U of such systems, where the real pole placement map is not surjective. It follows that for each system in U, there exists an open set of pole configurations which cannot be assigned by any real output feedback.

1. Introduction

Let $\mathbf{F}$ be one of the fields $\mathbf{C}$ (complex numbers) or $\mathbf{R}$ (real numbers). For fixed positive integers m,n,p, we consider matrices A,B,C and K with entries in $\mathbf{F}$, of sizes $n\times n$, $n\times m$, $p\times n$ and $m\times p$, respectively. The space Poly_n of monic polynomials of degree n with coefficients in $\mathbf{F}$ is identified with $\mathbf{F}^n$, using coefficients as coordinates. For fixed A,B and C with entries in $\mathbf{F}$, the *pole placement map* $\chi = \chi_{A,B,C}$ is defined as

$$\chi : \text{Mat}_{\mathbf{F}}(m\times p) \to \text{Poly}_n \simeq \mathbf{F}^n, \quad \chi(K) = \varphi_K,$$

where

$$\varphi_K(s) = \det(sI - A - BKC). \tag{1}$$

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This map will be called the real or complex pole placement map, depending on $\mathbf{F}$.

Our primary interest is in the real pole placement map. We briefly explain how it arises in control theory. (A general reference for this theory is [3], and a survey of the pole placement problem is [2]). A linear system with static output feedback is described by a system of equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= Ky. \end{aligned}$$

Here the state x, the input u and the output y are functions of a real variable t (time), with values in $\mathbf{R}^n$, $\mathbf{R}^m$, and $\mathbf{R}^p$, respectively, the dot denotes the derivative with respect to t, and A, B, C, K are real matrices of the sizes specified above. The matrix K is usually called a *gain* matrix, or a compensator. Eliminating u and y gives

$$\dot{x} = (A + BKC)x,$$

thus φ_K is the characteristic polynomial of the system. The pole placement problem is:

Given real A, B, C and a set of points $\{s_1, \ldots, s_n\}$ (listed with multiplicities) in $\mathbf{C}$, symmetric with respect to the real axis, to find real K such that the zeros of φ_K are exactly $s_1, \ldots, s_n$.

This amounts to finding real solutions of n algebraic equations with mp unknowns, the entries of K. For any dimensions m, n, p, there are systems (A, B, C) such that the map $\chi_{A,B,C}$ is not surjective (for example, if $B = 0$). So we consider generic systems (A, B, C). We say that for given m, n, p, the (real or complex) pole placement map is *generically surjective* if there exists an open dense set V in the space $\mathrm{Mat}_{\mathbf{F}}(n \times n) \times \mathrm{Mat}_{\mathbf{F}}(n \times m) \times \mathrm{Mat}_{\mathbf{F}}(p \times n)$, such that for $(A, B, C) \in V$, the map $\chi_{A,B,C}$ has dense image in $\mathbf{R}^n \simeq \mathrm{Poly}_n$. Thus "generic surjectivity" is a property of a triple of integers (m, n, p).

All topological terms in this paper refer to the usual topology.

We recall the known results on generic surjectivity. The case $\min\{m, p\} = 1$ is completely understood (see, for example, [2, 8]). From consideration of dimensions follows that $mp \geq n$ is a necessary condition of generic surjectivity, complex or real. If $mp > n$, then the real pole placement map is generically surjective [11, 7]. So from now on we restrict our attention to the case $n = mp$.

Theorem A [1] *For $n = mp$, the complex pole placement map is generically surjective. Moreover, it is a finite map of degree*

$$d(m,p) = \frac{1!2!\dots(p-1)!\,(mp)!}{m!(m+1)!\dots(m+p-1)!}.$$

The numbers $d(m,p)$ occur as the solution of the following problem of enumerative geometry: how many m-subspaces intersect mp given p-subspaces in $\mathbf{C}^{m+p}$ in general position? The answer $d(m,p)$ was obtained by Schubert in 1886 (see, for example, [6]).

One can deduce from Theorem A that the real pole placement map is generically surjective if $d(m,p)$ is odd. This number is odd if and only if one of the following conditions is satisfied [2]: a) $\min\{m,p\} = 1$, or b) $\min\{m,p\} = 2$ and $\max\{m,p\} + 1$ is an integral power of 2.

If $n = mp$, the real pole placement map was known not to be generically surjective in the following two cases: a) $m = p = 2$ [12], and b) $\min\{m,p\} = 2$, $\max\{m,p\} = 4$. For the case b), a rigorous computer assisted proof was given in [8]. It disproved an earlier conjecture, that a) is the only case when $n = mp$ and the real pole placement map is not generically surjective.

In this paper we show that infinitely many more cases exist.

Theorem 1 *If $n = mp$, $\min\{m,p\} = 2$, and $\max\{m,p\}$ is even, then the real pole placement map is not generically surjective.*

Our proof of Theorem 1 gives explicitly a system (A,B,C), and a point $w \in \mathrm{Poly}_n$, such that for any (A',B',C') close to (A,B,C), the real pole placement map $\chi_{A,B,C}$ omits a neighborhood of w.

In Section 2 we prove Theorem 1 using only elementary algebra, but the real reasons why this proof works are explained in Sections 3 and 4. In Section 3 we exhibit a wider class of counterexamples to pole placement, which includes those presented in Section 2. The content of Section 3 is based on a result from [4], whose rather complicated proof is outlined in Section 4. The word "circle" in this paper always means a circle on the Riemann sphere $\mathbf{CP}^1$, which corresponds to a circle or a line in the complex plane.

2. A class of linear systems and proof of Theorem 1

We begin with a well-known transformation of the expression (1). We factorize the rational matrix-function $C(sI-A)^{-1}B$, as

$$C(sI-A)^{-1}B = D(s)^{-1}N(s), \quad \det D(s) = \det(sI-A), \tag{2}$$

where D and N are polynomial matrix-functions of sizes $p \times p$ and $p \times m$, respectively. For the possibility of such factorization we refer to [3, Assertion 22.6]. Using (2), and the identity $\det(I-PQ) = \det(I-QP)$, which is true for all rectangular matrices of appropriate dimensions, we write

$$\begin{aligned} \varphi_K(s) &= \det(sI-A-BKC) = \det(sI-A)\det(I-(sI-A)^{-1}BKC) \\ &= \det(sI-A)\det(I-C(sI-A)^{-1}BK) \\ &= \det D(s)\det(I-D(s)^{-1}N(s)K) = \det(D(s)-N(s)K). \end{aligned}$$

This can be rewritten as

$$\varphi_K(s) = \det\left([D(s), N(s)]\begin{bmatrix} I \\ -K \end{bmatrix}\right). \tag{3}$$

It is convenient to extend φ to a map between compact manifolds. For this purpose, we allow an arbitrary $(m+p)\times p$ matrix L of rank p in (3) instead of

$$\begin{bmatrix} I \\ -K \end{bmatrix}.$$

A system is called *non-degenerate* if $\varphi_L \neq 0$ for every $(m+p)\times p$ matrix L of rank p. Such matrices are called equivalent, $L_1 \sim L_2$ if $L_1 = L_2U$ where $U \in GL_p(\mathbf{F})$. The set of equivalence classes is the Grassmannian $G_{\mathbf{F}}(p, m+p)$ which is a compact algebraic manifold of dimension mp. If $L_1 \sim L_2$, we have $\varphi_{L_1} = c\phi_{L_2}$, where $c \neq 0$ is a constant. The space of all non-zero polynomials of degree at most mp, modulo proportionality, is identified with the projective space $\mathbf{FP}^{mp}$, coefficients of the polynomials serving as homogeneous coordinates. This construction extends the pole placement map of a non-degenerate system to a regular map of compact algebraic manifolds

$$\chi : G_{\mathbf{F}}(p, m+p) \to \mathbf{FP}^{mp}, \tag{4}$$

where $\chi(L)$ is the proportionality class of the polynomial

$$\varphi_L(s) = \det\left([D(s), N(s)]\ L\right), \tag{5}$$

and L is an $(m+p)\times p$ matrix of rank p representing a point in $G_{\mathbf{F}}(p, m+p)$.

To prove Theorem 1, we set $p = 2$. This does not restrict generality in view of the symmetry of our problem with respect to the interchange of m and p, see, for example, [9, Theorem 3.3]. We consider a system (A_0, B_0, C_0) represented by

$$[D(s), N(s)] = \begin{bmatrix} s^{m+1} & s^m & \dots & s & 1 \\ (m+1)s^m & ms^{m-1} & \dots & 1 & 0 \end{bmatrix}. \tag{6}$$

The second row of $[D(s), N(s)]$ is the derivative of the first. The matrices A_0, B_0, C_0 can be recovered from $[D, N]$, by [3, Theorem 22.18]. Let $L = (a_{i,j})$. Introducing polynomials

$$f_{1,L}(s) = a_{1,1}s^{m+1} + a_{2,1}s^m + a_{3,1}s^{m-1} + \ldots + a_{m+2,1} \qquad \text{and}$$
$$f_{2,L}(s) = a_{1,2}s^{m+1} + a_{2,2}s^m + a_{3,2}s^{m-1} + \ldots + a_{m+2,2},$$

we can write (3) as

$$\varphi_L = f_{1,L}f'_{2,L} - f'_{1,L}f_{2,L} = W(f_{1,L}, f_{2,L}). \tag{7}$$

Thus for our system (A_0, B_0, C_0), the pole placement map becomes the *Wronski map*, which sends a pair of polynomials into their Wronski determinant. We say that two pairs of polynomials are equivalent, $(f_1, f_2) \sim (g_1, g_2)$, if $(g_1, g_2) = (f_1, f_2)U$, where $U \in GL_2(\mathbf{F})$. Equivalent pairs have proportional Wronski determinants. Equivalence classes of pairs of non-proportional polynomials of degree at most $m+1$ parametrize the Grassmannian $G_{\mathbf{F}}(2, m+2)$. A pair of complex polynomials will be called *real* if it is equivalent to a pair of real polynomials. The system represented by (6) is non-degenerate. This is a consequence of the well known fact that the Wronski determinant of two polynomials is zero if and only if the polynomials are proportional.

We will consider rational functions $f_L = f_{2,L}/f_{1,L}$. Equivalence relation on polynomial pairs defined above corresponds to the equivalence relation on rational functions: $f \sim g$ if $f = \ell \circ g$, where ℓ is a fractional-linear transformation. If r is a common factor of maximal degree of polynomials f_1 and f_2 then the roots of $W(f_1, f_2)$ are the critical points of the rational function f_2/f_1 plus the roots of r^2, all counted with multiplicity.

To prove Theorem 1, we use the following general result:

If, for some (m, n, p), there exists a real non-degenerate system (A_0, B_0, C_0) such that the real pole placement map χ_{A_0,B_0,C_0} in (4) is not surjective, then for these (m, n, p) the real pole placement map is not generically surjective.

Indeed, if χ_{A_0,B_0,C_0} omits one point w, it omits a neighborhood of w, because the image of a compact space under a continuous map is compact. Then for all (A, B, C) in a neighborhood of (A_0, B_0, C_0), the maps $\chi_{A,B,C}$ omit a neighborhood of w.

So, to prove Theorem 1, it is enough to find a non-zero real polynomial of degree at most $2m$ which cannot be represented as the Wronski determinant of a pair of real polynomials of degree at most $m + 1$.

Thus Theorem 1 is a corollary from

Proposition 1. *If $m \geq 2$ is an even integer, then the polynomial $w(s) = s(s^2 + 1)^{m-1}$ cannot be represented as the Wronski determinant of a pair of real polynomials of degree at most $m + 1$.*

Proof. The group $\mathrm{Aut}(\mathbf{CP}^1)$ of fractional linear transformations acts on the space $\mathbf{CP}^k$ of proportionality classes of non-zero polynomials of degree at most k by the following rule. Let

$$\ell(s) = \frac{as + b}{cs + d}, \quad ad - bc \neq 0$$

represent a fractional-linear transformation. For a polynomial $r(s)$, we put

$$\ell r(s) = (-cs + a)^k r \circ \ell^{-1}(s).$$

That this is indeed a group action, can be verified as follows. The space of proportionality classes of non-zero polynomials of degree at most k can be canonically identified with the symmetric power $\mathrm{Sym}^k(\mathbf{CP}^1)$, which is the set of unordered k-tuples of points in $\mathbf{CP}^1$. To each polynomial r one puts into correspondence its roots, counted with multiplicity, and the point ∞ with multiplicity $k - \deg r$. Then the action of $\ell \in \mathrm{Aut}(\mathbf{CP}^1)$ on such k-tuple is simply

$$(s_1, \ldots, s_k) \mapsto (\ell(s_1), \ldots, \ell(s_k)).$$

It is easy to verify that this action of $\mathrm{Aut}(\mathbf{CP}^1)$ extends to the space $G_{\mathbf{C}}(2, m + 2)$ of equivalence classes of polynomial pairs. Furthermore, this extended action is respected by the Wronski map:

$$W(\ell g_1, \ell g_2) = \ell W(g_1, g_2).$$

We use this fact to simplify the polynomial equation

$$W(g_1, g_2) = w, \quad w(s) = s(s^2+1)^{m-1} \tag{8}$$

which has to be solved to prove Proposition 1.

Consider the fractional-linear transformations

$$\ell(s) = \frac{1-is}{1+is}, \quad \ell^{-1}(s) = \frac{is-i}{s+1}.$$

We have $\ell : (0, i, \infty, -i) \mapsto (1, \infty, -1, 0)$, $\ell(\mathbf{R}) = \mathbf{T}$, the unit circle. Then

$$\ell w(s) = (s+1)^{2m} w \circ \ell^{-1}(s) \sim s^{m+1} - s^{m-1} = v(s),$$

where "$\sim$" means "proportional". Thus the equation (8) is equivalent to the equation

$$W(f_1, f_2) = v, \quad v(s) = s^{m+1} - s^{m-1}, \tag{9}$$

which we are going to solve now. Every solution (g_1, g_2) of (8) is equivalent to $(\ell^{-1} f_1, \ell^{-1} f_2)$, where (f_1, f_2) is a solution of (9).

For a polynomial r we denote by $\operatorname{ord} r$ the multiplicity of a root at 0. Replacing (f_1, f_2) by an equivalent pair, we can always achieve that

$$\deg f_1 \neq \deg f_2 \quad \text{and} \quad \operatorname{ord} f_1 \neq \operatorname{ord} f_2. \tag{10}$$

Then

$$\begin{aligned} m-1 &= \operatorname{ord} v = \operatorname{ord} f_1 + \operatorname{ord} f_2 - 1, \\ m+1 &= \deg v = \deg f_1 + \deg f_2 - 1. \end{aligned}$$

So

$$(\deg f_1 - \operatorname{ord} f_1) + (\deg f_2 - \operatorname{ord} f_2) = 2,$$

and we have two cases to consider.

Case 1. $\deg f_1 - \operatorname{ord} f_1 = \deg f_2 - \operatorname{ord} f_2 = 1$. In this case,

$$f_1(s) = as^{k+1} + bs^k, \quad f_2(s) = cs^{l+1} + ds^l,$$

where a, b, c, d are non-zero complex numbers, $k + l = m$, and $k < l$ (without loss of generality). For the Wronskian $W(f_1, f_2)$, we obtain

$$ac(l-k)s^{m+1} + (ad(l-k-1) + bc(l-k+1))\, s^m + bd(l-k)s^{m-1}.$$

Substituting this to (9) gives

$$ac = -bd \quad \text{and} \quad ad(l-k-1) + bc(l-k+1) = 0.$$

Solving these equations, we conclude that the rational function $f = f_2/f_1$ has the form

$$f(s) = \lambda s^q \frac{s-\mu}{s+1/\mu}, \quad \text{where} \quad q = l-k \geq 2,\ \lambda \in \mathbf{C}^*,\ \mu = \pm\sqrt{\frac{q+1}{q-1}}.$$

Notice that the image of the unit circle under this f *cannot* belong to a circle. Indeed, if we suppose that this image belongs to a circle S, then the points $0 = f(0)$ and $\infty = f(\infty)$ are symmetric with respect to S, so S is centered at 0, and thus $f \circ \sigma = \text{const} \cdot \sigma \circ f(s)$, where $\sigma(s) = 1/\overline{s}$, which is evidently not true.

Case 2. $f_1(s) = cs^l$, $f_2(s) = as^{k+1} + bs^{k-1}$, where a, b, c are non-zero complex numbers, $l \neq k \pm 1$. Computing $W(f_1, f_2)$ gives

$$ac(k-l+1)s^{k+l} + bc(k-l-1)s^{l+k-2},$$

and substitution to (9) gives $l + k = m + 1$, $(k-l+1)ac = -(k-l-1)bc$. This implies that the rational function $f = f_2/f_1$ has the form

$$f(s) = \lambda(s^{q+1} - \mu s^{q-1}), \quad \text{where} \quad q = k-l,\ \lambda \in \mathbf{C}^*,\ \mu = \frac{q+1}{q-1}.$$

From our assumption that m is even follows that $q = k - l$ is odd, thus $q \neq 0$. Furthermore, $q \neq \pm 1$ in view of (10), in particular, $\mu \neq 0$. Under these conditions, the image of the unit circle under f is not contained in any circle. Indeed, $f(\{0, \infty\}) = \{0, \infty\}$, and the same argument as in Case 1 works.

Now we can complete the proof of Proposition 1. Suppose that (8) has a real solution (g_1, g_2). Then the rational function $g = g_2/g_1$ is real. As ℓ^{-1} maps the unit circle onto the real line, we conclude that $g \circ \ell^{-1}$ is real on the unit circle. This implies that f, which is equivalent to $g \circ \ell^{-1}$, maps the unit circle into a circle, but we know from the explicit forms of f obtained in Cases 1 and 2 that this is not so. Thus no solution of (8) can be real. □

3. Rational functions with critical points on a circle

In this section we use

Proposition 2 [4] *If all critical points of a rational function f belong to a circle S, then f maps S into a circle.*

This result permits to prove the following generalization of Proposition 1:

Proposition 3 *Let $m \geq 2$ be even, S a circle orthogonal to the real line, and w a real polynomial whose roots belong to S. Suppose that one of the two conditions holds:*
a) $\infty \notin S,\ \deg w = 2m$, and both points of $S \cap \mathbf{R}$ are simple roots of w, or
b) $\infty \in S,\ \deg w = 2m-1$, and the only point in $S \cap \mathbf{R}$ is a simple root of w.
Then w cannot be represented as the Wronski determinant of a pair of real polynomials of degrees at most $m+1$.

Proof of Proposition 3. We give the proof only for the case a). Case b) can be treated similarly, or derived from case a) using the action of $\mathrm{Aut}(\mathbf{CP}^1)$ described in the proof of Proposition 1. Thus we assume

$$\deg w = 2m. \tag{11}$$

Suppose that

$$W(f_1, f_2) = w, \tag{12}$$

where w satisfies the conditions of Proposition 3, and (f_1, f_2) is a pair of real polynomials of degree at most $m+1$.

We define $f = f_1/f_2$, and claim that

$$\deg f \quad \text{is odd.} \tag{13}$$

To prove the claim, we may assume that

$$\deg f_1 \leq \deg f_2 - 1, \tag{14}$$

replacing if necessary the pair (f_1, f_2) by an equivalent pair. Let r be a common factor of f_1 and f_2 of maximal degree. We can choose r to be real. Each root of r is a root of w of even multiplicity. As w has only simple roots on the real line, we conclude that r has no real roots. So

$$\deg r \quad \text{is even.} \tag{15}$$

By assumption, we have $\deg f_2 \leq m+1$. Combined with (11) and (14) this gives

$$2m = \deg w = \deg f_1 + \deg f_2 - 1 \leq 2\deg f_2 - 2 \leq 2m,$$

so

$$\deg f_2 = m+1, \quad \text{and} \quad \deg f_1 = \deg f_2 - 1. \tag{16}$$

Then $\deg f = \deg f_2 - \deg r = m + 1 - \deg r$ is odd, because m is even by assumption, and $\deg r$ is even by (15).

As we suppose that f_1 and f_2 are real, the rational function $f = f_2/f_1$ is real. The second equation in (16) implies that ∞ is not a critical point of f. The critical points of f in $\mathbf{C}$ are contained in the set of roots of w, so all critical points of f belong to S. By Proposition 2, the image $f(S)$ is contained in a circle S'. This circle S' is symmetric with respect to $\mathbf{R}$, because f is real, and S is symmetric with respect to $\mathbf{R}$. So either $S' = \overline{\mathbf{R}}$ or S' is orthogonal to R. The second possibility is excluded, because S is orthogonal to $\mathbf{R}$, and f has simple critical points at the intersection $S \cap \overline{\mathbf{R}}$. Thus $S = \overline{\mathbf{R}}$. Let $\sigma(s) = \overline{s}$ and τ be the reflections with respect to $\overline{\mathbf{R}}$ and S, respectively. As $f(\overline{\mathbf{R}}) \subset \overline{\mathbf{R}}$, and $f(S) \subset \overline{\mathbf{R}}$, we have $f \circ \sigma = \sigma \circ f$ and $f \circ \tau = \sigma \circ f$, so

$$f \circ \sigma \circ \tau = f. \tag{17}$$

The composition $\ell = \sigma \circ \tau$ of reflections in two orthogonal circles is a fractional-linear transformation with two fixed points $\overline{\mathbf{R}} \cap S$, and $\ell \circ \ell = \mathrm{id}$. Now (17) implies that a rational function h of degree 2, whose critical set coincides with $S \cap \overline{\mathbf{R}}$, is a right compositional factor of f, that is $f = g \circ h$, where g is a rational function. This contradicts (13). □

4. Outline of the proof of Proposition 2.

Taking $S = \mathbf{R}$ in Proposition 2, we obtain: *Every rational function whose critical points are real is equivalent to a real rational function.* This implies something opposite to Proposition 1, namely: the pole placement problem for a system (6), with distinct *real* points $\{s_1, \ldots, s_{2m}\}$ always has $d(m,2)$ real solutions. This is a special case of the B. and M. Shapiro conjecture, [9]. Systems similar to (6) were recently used to prove that the real pole placement map is generically surjective when $m+p$ is odd [4]. This shows that Theorem 1 detects all cases when $p = 2$ and the real pole placement map is not generically surjective.

Now we explain the ideas behind the proof of Proposition 2. It is derived from Theorem A and

Proposition 4 [4] *Given $2m$ distinct points on a circle S, there exist at least $d(2,m)$ non-equivalent rational functions of degree $m+1$ having these critical points, and mapping S into a circle.*

Theorem A with $p=2$ implies: *Given $2m$ points on a circle S, there exist at most $d(2,m)$ classes of rational functions of degree $m+1$ with these critical points.* Comparing this with Proposition 4, we conclude that all these $d(2,m)$ classes contain functions mapping S into a circle. This proves Proposition 2 in the case when all critical points are simple. An additional approximation argument is necessary to prove Proposition 2 when multiple critical points are present, [4, Section 7].

Now we outline the proof of Proposition 4. Without loss of generality, we may assume that $S = \mathbf{T}$, the unit circle. The word "symmetry" in what follows always refers to the symmetry with respect to $\mathbf{T}$. Let R be the class of all rational functions f of degree $m+1 \geq 3$, such that $f(\mathbf{T}) \subset \mathbf{T}$, $f(1) = 1$, $f'(1) = 0$, all critical points of f are simple and belong to $\mathbf{T}$. We first describe a convenient parametrization of a subclass of properly normalized functions in R. We consider a *net*, $\gamma(f) = f^{-1}(\mathbf{T})$. It coincides with the 1-skeleton of a cell decomposition of $\mathbf{CP}^1$, whose 2-dimensional cells are components of $f^{-1}(\mathbf{CP}^1 \backslash \mathbf{T})$, and 0-dimensional cells (vertices) are the critical points of f. It is easy to see that this cell decomposition has the following properties:

(i) γ is symmetric with respect to $\mathbf{T}$ and contains $\mathbf{T}$,

(ii) all $2m$ vertices belong to $\mathbf{T}$ and have order 4,

(iii) the point $1 \in \mathbf{T}$ is a vertex.

Such cell decompositions are uniquely determined by their 1-skeletons. Two nets γ_1 and γ_2 are called *equivalent* if there exists an orientation-preserving symmetric homeomorphism $\eta : \mathbf{CP}^1 \to \mathbf{CP}^1$, fixing the point 1, and such that $\eta(\gamma_1) = \gamma_2$. We *label* each edge e of a net $\gamma(f)$ by a non-negative number $p(e)$, the length of the arc $f(e) \subset \mathbf{T}$. Then

$$\sum_{e \in \partial G} p(e) = 2\pi \quad \text{for every 2-dimensional cell } G. \tag{18}$$

A *labeled net* is a 1-skeleton of a cell decomposition of $\mathbf{CP}^1$ satisfying N1-N3, equipped with a non-negative symmetric function p on the set of edges,

satisfying (18). If p is strictly positive, we call the labeling non-degenerate. Using the Uniformization Theorem (for the sphere) we prove: *To each labeled net corresponds a rational function $f_{\gamma,p}$, such that for non-degenerate p we have $\gamma \sim f_{\gamma,p}^{-1}(\mathbf{T})$, and $p(e) = \text{length} f(e)$, for every edge e of $f_{\gamma,p}^{-1}(\mathbf{T})$.*
This result gives a parametrization of a properly normalized subclass of R by classes of labeled nets $([\gamma], p)$, where $[\gamma]$ is a class of equivalent nets, and p a non-degenerate labeling. This parametrization has an advantage that it separates topological information about f, described by the class $[\gamma]$, from the continuous parameters p, in such a way that the space of continuous parameters is topologically trivial. Indeed, the space of all non-degenerate labelings of a fixed net is a convex polytope Λ_γ, described by equations (18), the symmetry conditions, and the condition $p > 0$. The set of all labelings of γ is the closure $\overline{\Lambda}_\gamma$ of Λ_γ. For a fixed net γ and every $p \in \Lambda_\gamma$ we consider the critical set of $f_{\gamma,p}$, which is a sequence of $2m$ distinct points on the unit circle containing the point 1. We denote the space of all such sequences by Σ_γ. It can be identified with an open convex polytope of the same dimension as Λ_γ. Thus for every γ we have a map $\Phi_\gamma : \Lambda_\gamma \to \Sigma_\gamma$. We want to show that it is surjective, using a "continuity method," going back to Poincaré and Koebe. First we extend Φ_γ to a continuous map of the closed polytopes

$$\Phi_\gamma : \overline{\Lambda}_\gamma \to \overline{\Sigma}_\gamma.$$

Rational functions $f_{\gamma,p}$ corresponding to $p \in \partial\Lambda_\gamma$ either have multiple critical points, or degree strictly less than $m + 1$. The proof of continuity of the extended map Φ_γ uses classical function theory: Picard's and Montel's theorems. The surjectivity of Φ_γ is derived from the careful analysis of its boundary behavior. The purpose of this analysis is to show that full preimages of the closed faces of $\overline{\Sigma}_\gamma$ are topologically trivial, that is they have the same homology groups as a one-point space. In particular, these preimages are non-empty and connected. As the preimages of the closed faces can be rather complicated, this is done in several steps. First we describe the preimages of *open faces*, which are much simpler because they are convex. Then we consider chains $A_1, A_2, \ldots, A_k$ of open faces of $\overline{\Sigma}$, such that $\overline{A}_1 \supset \overline{A}_2 \supset \ldots \supset \overline{A}_k \neq \emptyset$, and show that, for each such chain, the closures of preimages of open faces have non-empty intersection:

$$\overline{\Phi_\gamma^{-1}(A_1)} \cap \ldots \cap \overline{\Phi_\gamma^{-1}(A_k)} \neq \emptyset.$$

This is enough to deduce that the preimages of closed faces are topologically trivial. Once this is established, we use the following fact.

Lemma 1 *Let $\Phi : \overline{\Lambda} \to \overline{\Sigma}$ be a continuous map of closed convex polytopes of the same dimension. If the preimage of every closed face of $\overline{\Sigma}$ is (non-empty and) topologically trivial, then Φ is surjective.*

Thus for every class of nets γ and every critical set in Σ_γ we have a rational function $f \in R$, having this critical set, and such that $\gamma \sim f^{-1}(\mathbf{T})$. Since equivalent functions have equivalent nets, there are at least as many equivalence classes of functions in R, sharing a given critical set, as the number of classes of nets with $2m$ vertices. So the proof of Proposition 4 is completed by simple combinatorics:

Lemma 2 *The number of classes of nets on $2m$ vertices is $d(m, 2)$, the d-th Catalan number.*

For this we refer to [10, Exercise 6.19 n].

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