MA 511 Final Exam A. Eremenko

NAME____

1. Let

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -4 & -1 & 3 \\ -2 & 0 & 1 \end{pmatrix}.$$

a) Factor $A = LU$, where L and U are lower and upper triangular, respectively.

b) Find dimensions of the four fundamental subspaces.

c) Find a basis for the column space of A.

d) Find a basis for the nullspace of A.

2. What can the eigenvalues of a 4×4 permutation matrix be? Give an example for each possibility. Are these matrices diagonalizable?

3. a) Find the rank and eigenvalues of the following matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

b) Do the same for such matrices of arbitrary dimension (A consists of 1's, and B has the checkerboard pattern. The size of B is an even number.)

4. Let $A = (a_{i,j})$ be a square matrix with $a_{i,j} = 1$ if $j \geq i$ and $a_{i,j} = 0$ otherwise. Find the eigenvalues, eigenvectors and the Jordan form of A.

5. Let $\mathcal{P}_4$ be the space of all polynomials $p(x)$ of degree at most 4. Let V and W be two subspaces of $\mathcal{P}_4$ consisting of all polynomials satisfying $p(0) = 0$ and $p(1) = 0$, respectively. Find dimensions of V, W, the intersection $W \cap V$ and the span $V + W$.

6. In the space of polynomials of degree at most 2, consider a linear transformation which sends $p(x)$ to $xp'(x) - p(x)$.
a) Choose your favorite basis and write the matrix of this transformation in this basis.
b) What are the eigenvalues of this transformation?

7. Write the least squares solution to $A\,\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

8. Let $A = \begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix}$.
For which real s this matrix is positive definite?

9. Solve the differential equation

$$\frac{d\mathbf{u}}{dx} = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} \mathbf{u} \quad \text{with} \quad \mathbf{u}(0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}.$$

Your answer should not include complex numbers.

10. Let Q be a 3×3 orthogonal matrix which satisfies $Q^3 = I$ and has an eigenvector $(1, 1, 1)^T$. How many such matrices exist?

11. Consider the standard coordinate system e_1, e_2, e_3 in R^3. Let R_1 be the rotation around the e_2-axis, by 90^o anticlockwise, when looking from the origin in the positive direction of the e_2 axis. Let R_2 be the rotation around the e_3-axis, by 90^o anticlockwise, when looking from the origin in the positive direction of the e_3-axis. Find the axis of rotation and the absolute value of the angle of rotation of the product $R_1 R_2$.