Math 530, Spring 2024. Practice problems for the midterm.

Midterm will contain 5 problems, and last 50 minutes.

1. Suppose that the radius of convergence of the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is equal to R. Find the radii of convergence of the following series:

a) $\displaystyle\sum_{n=0}^{\infty} c_n^m z^n \quad (m \quad \text{is a positive integer}),$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{c_n}{1+|c_n|} z^n.$

2. Denote the radii of convergence of the series

$$\sum_{n=0}^{\infty} a_n z^n, \quad \sum_{b=0}^{\infty} b_n z^n, \quad \sum_{n=0}^{\infty} (a_n + b_n) z^n, \quad \sum_{n=0}^{\infty} a_n b_n z^n$$

by $R_a,\ R_b,\ R_{a+b}$ and R_{ab} respectively. Prove

$$R_{a+b} \geq \min\{R_a, R_b\}, \quad \text{and} \quad R_{ab} \geq R_a R_b.$$

Give examples of strict inequalities.

3. Let $f(z) = u(x,y) + iv(x,y),\ z = x + iy$, be a function which is **C**-differentiable at a point $z_0 = x_0 + iy_0$. Prove the formulas:

$$\begin{aligned} f'(z_0) &= u_x(x_0,y_0) + iv_x(x_0,y_0) && (1)\\ &= v_y(x_0,y_0) - iu_y(x_0,y_0) && (2)\\ &= u_x(x_0,y_0) - iu_y(x_0,y_0) && (3)\\ &= v_y(x_0,y_0) + iv_x(x_0,y_0). && (4) \end{aligned}$$

and

$$|f'(z_0)|^2 = u_x^2(x_0,y_0) + u_y^2(x_0,y_0).$$

4. Find all solutions of the equation

$$(z+1)^5 = 2z^5.$$

5. a) Show that the product of two harmonic functions uv is harmonic if and only if there are constants α, β such that αu and βv satisfy the Cauchy-Riemann equations, where k is a constant. Geometrically this means that their gradients are perpendicular at every point.

b) Let f be a non-constant analytic function in a region D. Can $|f|$ or $|f|^2$ be harmonic in D?

6. Does there exist an analytic function whose real part is

$$u(x,y) = (e^y + e^{-y}) \sin x.$$

If the answer is negative, explain why. If positive, find this analytic function.

7. Let f be an analytic function in a region symmetric with respect to the real line. a) Prove that $g(z) := \overline{f(\overline{z})}$ is also analytic. b) Can $f(\overline{z})$ be analytic for non-constants analytic f?

8. Let P be a real polynomial whose all roots belong to the left half-plane. Prove that the coefficients of P are strictly positive.

(You may use the Fundamental theorem of Algebra which says that every non-constant polynomial factors into polynomials of degree 1).

9. Factor $x^4 + 1$ into two real quadratic polynomials.

10. Let

$$f(z,w) = \frac{z-w}{1-\overline{w}z}, \quad \text{where} \quad 1 - \overline{w}z \neq 0.$$

a) Prove that when $|z| < 1$ and $|w| < 1$ we have $|f(z,w)| < 1$.

b) Prove that if $|w| = 1$ then $|f(z,w)| = 1$ for all z such that $1 - \overline{w}z \neq 0$.

11. Find the general form of a linear-fractional map of the right half-plane onto itself.

12. Find a conformal map of the region

$$D = \{z : |z| < 1,\ |z - 1/2| > 1/2\}$$

onto the upper half-plane.

13. Does there exists an analytic function which maps

a) the unit disk onto the complex plane surjectively,

b) the complex plane onto the unit disk surjectively.

In each case either prove that it does not exist, or give an example.

14. Let D be a convex region, and f a function analytic on D with the property that $|f'(z)| < 1,\ z \in D$. Prove that $g(z) = z + f(z)$ is injective in D.

15. Is Rolle's theorem true for complex functions? Suppose that f is analytic in a convex region, for example in $\mathbf{C}$. If z_1 and z_2 are two points in D, and $f(z_1) = f(z_2) = 0$, does it follow that there is a point $z_3 \in [z_1, z_2]$ such that $f'(z_3) = 0$? Does it follow that there exists such a point $z_3 \in \mathbf{C}$ that $f'(z_3) = 0$?

15a. Suppose that all roots of a polynomial f belong to some half-plane. Prove that all roots of f' belong to the same half-plane.

(Hint: consider

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{d} \frac{1}{z - z_n},$$

where $z_1, \ldots, z_d$ are the roots.)

Derive the Gauss–Lucas theorem: all roots of the derivative of a polynomial belong to the convex hull of the roots of this polynomial.

16. Prove that a reflection in a circle maps circles to circles. (Here the word "circle" means "a circle or a line".)

17. Let A and B be two disjoint (round) circles. Start with a circle C_1 which is tangent fo A and B. Draw a circle C_2 which is tangent to A, B, C_1. Then draw a circle $C_3 \neq C_1$ which is tangent to A, B, C_2, and continue this procedure. So that C_{n+1} is tangent to A, B, C_n but is different from C_{n-1}. Either this procedure will continue indefinitely, and give infinitely many distinct circles, or you arrive at an already existing circle at some step.

Prove that the outcome (the total number of circles or ∞) does not depend on the initial circle C_1, that is the result is completely determined by A and B.

18. Evaluate the integrals

$$\int_{|z|=2} \frac{\cos z}{(z-1)^3} dz \quad \text{and} \quad \int_{|z|=2} \frac{\cos z}{(z-1)(z-5)} dz.$$

19* (this is somewhat more difficult) Recall that a functions f is called regular in D if for every $a \in D$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \tag{5}$$

where the series converges to f in *some* disk around a.

Suppose that we do not know anything about integrals, Cauchy theorem and Cauchy formula. Prove directly that if f is regular in $|z| < R$, then the radius of convergence of its power series at 0 is at least R.

Hint: solve problem 20 first, and use the result.

20* Let f be a convergent power series as in (5) in the previous problem, and let R be its radius of convergence. Without using integrals, Cauchy's theorem or Cauchy Integral formula, prove that

$$|f(a)| = |c_0| \leq \max_{|z-a|=r} |f(z)|, \quad \text{for every} \quad r \in (0, R).$$

Can you prove the full Cauchy inequalities

$$|c_n| \leq \max_{|z-a|=r} |f(z)|/r^n \quad ?$$

Can you upgrade your proof to the proof of the Maximum Modulus Principle?

Also you may solve problems from the Ahlfors book which were not assigned as a HW.