MA 511 Practice exam

In problems 1-3 you do not have to show work. In problems 4-6 you should show work.

1. Circle the letters corresponding to statements which are true for any two square matrices A and B such that $AB = 0$:

A. Either $A = 0$ or $B = 0$.

B. $C(A) \subset N(B)$.

C. $C(A^T) \supset N(B^T)$.

D. $BA = 0$.

E. Either $A = 0$ or B is singular.

Ans.: E.

2. Which of the following is a subspace of the vector space $C[0,1]$ of continuous functions on $[0,1]$? Circle the letters corresponding to correct answers.

A. Functions with $f(1) = 0$.

B. Functions with $f(0) = 1$.

C. Functions with $\int_0^1 f(x)\,dx = 0$.

D. Functions with $f(0) = f(1)$.

E. Functions with $f(0) \leq f(1)$.

Ans.: A,C,D.

3. Which of the following is a linear transformation T of the space of 3×3 matrices? Circle the letters corresponding to correct answers.

A. $T(A) = U$ where U is the row echelon form of A.

B. $T(A) = A^{-1}$.

C. $T(A) = A^T$.

D. $T(A) = P$ where $p = Pb$ is the orthogonal projection of b to $C(A)$.

E. $T(A) = E_{2,1}(3)A$ where $E_{2,1}(3)$ is an elementary matrix.

Ans.: C,E.

4. Let

$$A = \begin{pmatrix} 1 & 2 & 2 \\ -6 & -11 & -12 \\ 1 & -4 & 3 \end{pmatrix}$$

a) Factor $A = L\,U$.

b) Find A^{-1}.

5. Let T be the transformation of the space $\mathbf{P}_2$ of polynomials of degree 2 defined as $T(p(x)) = p(x) - p(1)$.

a) Write the matrix of T in the basis 1, x, x^2 of $\mathbf{P}_2$.

b) Find the rank of T.

c) Find the fundamental subspaces $C(T)$ and $N(T)$.

Ans.: b) The rank is 2. c) $N(T)$ consists of constant polynomials. $C(T)$ consists of all polynomials with the property $p(1) = 0$.

6. Find the projection of $b = (10, 20, 30)^T$ onto the column space of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 2 \end{pmatrix}.$$

7. Find the distance from the origin to the plane $x + y + 2z = 3$.

Hint. We learned how to find a distance from a vector to a subspace. But this plane is not a subspace. To reduce the problem to the known problem, set $x = x_1, y = x_2 + 1, z = x_3 + 1$. Then our equation becomes

$$x_1 + x_2 + 2x_3 = 0, \tag{1}$$

which is a subspace, and the vector $(x, y, z)^T = (0, 0, 0)^T$ becomes $b := (0, -1, -1)^T$. It is evident that our transformation preserves all distances.

To apply the projection formulas, we find a basis in the subspace (1), for example $(-1, 1, 0)^T$ and $(-2, 0, 1)^T$ is a basis, and project the vector $b = (0, -1, -1)^T$ onto the column space of

$$A = \begin{pmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The projection is

$$p = A(A^T A)^{-1} A^T b$$

and the distance is $\|b - p\|$.

Another method: find an orthogonal basis in this plane (1): first find a basis $(-2, 0, 1)^T$, $(0, -2, 1)^T$ and then apply one step of the orthogonalization process to it. We find an orthogonal basis $q_1 = (-2, 0, 1)$, $q_2 = (-2/5, -2, 4/5)$. Then the square of the distance is

$$\|b - c_1 q_1 - c_2 q_2\|^2,$$

where

$$c_1 = \frac{b^T q_1}{q_1^T q_1}, \quad c_2 = \frac{b^T q_2}{q_2^T q_2}.$$

8. Is it true that for every square matrix we have

$$r(A) = r(A^T A) = r(AA^T)$$

where $r(.)$ is the rank?

If true, explain why; if not, give a counterexample.

Hint: Consider $N(A)$ and $N(A^T A)$.

Ans.: This is true, even for rectangular matrices. To show this, perform these steps.

1. $N(A) = N(A^T A)$. It is clear that $N(A) \subset N(A^T A)$. To show that also $N(A^T A) \subset N(A)$, consider a vector $x \in N(A^T A)$. Let $y = Ax$. Then $y \in C(A)$ and $y \in N(A^T)$. But $C(A) \perp N(A^T)$, because $C(A) = R(A^T) \perp N(A^T)$. So $y = 0$ that is $x \in N(A)$. (A similar argument used in classs in the derivation of projection formula. It is also in the book).

2. Suppose that A is $m \times n$. Then $A^T A$ is $n \times n$ while AA^T is $m \times m$. Now we have

$$r(A) = n - \dim N(A) = n - \dim N(A^T A) = r(A^T A). \tag{2}$$

And similarly

$$r(A^T) = m - \dim N(A^T) = r(AA^T), \tag{3}$$

where we applied step 1 to A^T.

3. But we also know that $r(A) = r(A^T)$, so (2) and (3) give the required equality

$$r(A) = r(A^T A) = r(AA^T).$$