Math 520, Spring 2007, Practice exam

1. Let f be a (2π)-periodic function such that $f(x) = xe^{-x}$ for $-\pi < x < \pi$. Suppose that $f(x) = g(x) + h(x),\ -\pi < x < \pi$ where

$$g(x) = \sum_{n=0}^{\infty} a_n \cos nx, \quad \text{and} \quad h(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Find a_0, g and h. Compute $h(\pi)$.

2. a) Find the Fourier series of the 2π-periodic extension of the function $f(x) = x,\ -\pi < x < \pi$.

b) Find the sum of this series at the points $x = \pm\pi$.

c) Using the fact that $(x^3)'' = 6x$, and a), find the Fourier series of the 2π-periodic extension of $f(x) = x^3,\ -\pi < x < \pi$.

3. Find the best approximation in $L^2(0, \pi)$ to the function $f(x) = \cos x$ by a linear combination of functions $\sin x,\ \sin 2x$ and $\sin 3x$.

4. Solve the equation $u_t = u_{xx} + e^{-t} \sin x$ for $0 \leq x \leq \pi$ and $t \geq 0$ with the boundary conditions $u(0, 1) = u(\pi, t) = 0$ and initial condition $u(x, 0) = \sin 2x$.

5. Find the eigenvalues and eigenfunctions for the following Sturm-Liouville problem on $[0, 1]$:

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0.$$

6. Consider the wave equation on the half-line $[0, +\infty)$:

$$u_{tt} = u_{xx},$$

The boundary conditions are $u(0, t) = 0$ and $u(+\infty, t) = 0$ The initial condition is $u_t(x) = 0,\ t \geq 0$ and $u(x, 0) = 1,\ 1 \leq x \leq 2$ and zero for the rest of x. Make pictures of the graph of $u(x, t)$ as a function of x for the following moments of time: $t = 1, t = 1.5, t = 2, t = 3, t = 4$.