Practice problems for the final

You should be able to:
– describe the set of solutions of any system of linear equations
– determine whether a system of vectors is linearly independent
– find a span of any given system of vectors
– find a basis in any given subspace
– write the matrix of a linear transformation with respect to a given basis
– find the kernel and image of any linear transformation
– find a projection of any vector on any given subspace
– find the distance between a vector and a subspace
– construct an orthonormal basis in any given subspace
– evaluate any given determinant of moderate size
– find the inverse martix by row operations and using determinants
– find eigenvalues and eigenspaces of any given linear transformation
– to determine whether a matrix is diagonalizable and if this is so, to diagonalize it.
– find a basis consisting of eigenvectors
– find the exponential of any given matrix
– solve linear difference equations with constant coefficients.

Below are some typical problems selected from the old exams.

1. Find the relation between $\det(\exp(A))$ and $\mathrm{tr}(A)$ for diagonalizable matrices A.

2. Let P_2 be the space of all polynomials of degree at most 2. Consider the linear transformation $T : P_2 \to P_2$ given by $T(f) = 2f' - f''$.
a) Find the matrix of T with respect to the basis $\{1, t, t^2\}$.
b) Is this transformation diagonalizable?

3. Let

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

a) Find the image of A.
b) Find an orthonormal basis in the image of A.
c) Find the projection of b onto the image of A.
c) Find the distance from b to the image of A.

4. Consider the following set of vestors in R^3:

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ a \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix} \right\}.$$

a) What relation(s) should a, b, c satisfy for S to be linearly independent?
b) For which a, b, c the set S is orthogonal?

5. Consider the following system of equations

$$\begin{array}{ccccccc} x & + & 2y & + & 6z & = & 2 \\ & & y & + & 2mz & = & 0 \\ mx & & & + & 2z & = & 1 \end{array}.$$

For which values of m does this system have
a) no solutions ?
b) infinitely many solutions ?
c) a unique solution ?

6. Is it true that $\text{rank}(A) = \text{rank}(A^T A)$ for every matrix A, not necessarily square ? (If yes, explain why, if no, give an example).

7. Solve the difference equation

$$a_{n+2} = 5a_{n+1} - 5a_n, \quad a_0 = 7, \quad a_1 = 17.$$

8. Is it true that every $m \times n$ matrix of rank one is a product of a $m \times 1$ matrix and a $1 \times n$ matrix, that is

$$A = (\text{column}) \times (\text{row}) \quad ?$$

(If yes, explain why, if no, give an example. If this looks too hard, consider the 2×2 case.)

9. Let A be a 4×4 matrix, and $\det(A) = -2$. Find

$$\det(AA^T), \quad \det(A^3), \quad \det(-2A), \quad \det(A^{-1}).$$

10. Find the values of a, b and c such that

$$A = \begin{bmatrix} 3 & a & b \\ 0 & -4 & c \\ 0 & 0 & 3 \end{bmatrix}$$

is diagonalizable.