THE MODIFIED CARTAN CONJECTURE

ALEXANDRE EREMENKO, ZONGBEN XU, AND TENG ZHANG

ABSTRACT. In 1928 H. Cartan stated a conjecture about holomorphic curves in $\mathbb{P}^n$ parametrized by the unit disc, omitting $n+2$ hyperplanes in general position. He proved it for $n = 2$. In 1996 the first-named author constructed counterexamples for all $n \geq 3$, and proposed a modified form of Cartan's conjecture which he proved for $n = 3$. In this paper we prove this modified conjecture in all dimensions. As a byproduct we prove a quantitative version of the classical theorem that analytic functions are linearly dependent iff their Wronskian determinant is zero. We interpret our results in terms of the Kobayashi metric on the complement of $n+2$ hyperplanes, and on smooth algebraic varieties in the algebraic torus.

1. Introduction

Emile Borel [Bor97] proved that the identity

$$f_1 + \ldots + f_p = 0$$

for zero-free entire functions can hold only for trivial reasons: when the set $\{f_1, \ldots, f_p\}$ can be partitioned into disjoint groups such that within each group all functions are constant multiples of one of them, and the sum of all functions in each group is zero. It follows from the last property that each group contains at least two elements.

The case $p = 3$ is equivalent to Picard's theorem [Pic79]. Indeed, if f_1, f_2, f_3 are zero-free entire functions with zero sum, then $f = -f_1/f_3$ is entire and omits 0 and 1. Conversely, every entire function f omitting 0 and 1 gives the zero-free triple $(-f, f-1, 1)$ with zero sum. Each group in Borel's theorem contains at least two elements, so there is only one group when $p = 3$. Thus all f_j are proportional to one of them, so f is constant.

So Borel's theorem is a generalization of Picard's theorem to holomorphic curves in projective space, parametrized by the complex plane.

Andre Bloch was promoting the philosophical principle that "Nihil est in infinito quod non prius fuerit in finito". When applied to function theory this is usually interpreted as *every condition which implies that a meromorphic function in the plane is constant, when imposed on a family of meromorphic functions in an arbitrary region, must imply normality (precompactness) of this family.* The prototype is Montel's normality criterion [Mon27] which says that a family of holomorphic functions omitting 0 and 1 is normal in the spherical metric. For a general discussion of Bloch's Principle we refer to the original exposition [Blo26a] and to the modern survey [Ber06]. For extensions to holomorphic curves we mention [Yam26].

Bloch [Blo26] applied his principle to Borel's theorem, and obtained some partial results, some of them only heuristic, without complete proofs. Then this question was studied by

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Henri Cartan in his thesis [Car28]. To state Cartan's result we use the following notation and definition.

We write $D(r) = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D} = D(1)$, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. By $\mathcal{O}(U)$ we denote the ring of holomorphic functions on an open set U with the topology of uniform convergence on compact subsets. A holomorphic function without zeros will be called a *unit*, in accordance with the terminology of ring theory. For $p \geq 3$, we consider sequences of p-tuples of units $(f_{1,n}, \dots, f_{p,n})_{n=1}^{\infty}$ on $\mathbb{D}$ such that

$$f_{1,n} + \cdots + f_{p,n} = 0, \quad n = 1, 2, \dots \tag{1.1}$$

The appropriate counterpart for a group of proportional functions is the following notion, introduced in [Ere96b].

Definition 1.1. *Let $\Omega \subset \mathbb{D}$ be a region. A subset $I \subseteq \{1, \dots, p\}$ is called a C-*class *on Ω if there is an index $k \in I$ such that the quotients $f_{j,n}/f_{k,n}$, $j \in I$, are locally uniformly bounded on Ω and*

$$\sum_{j \in I} \frac{f_{j,n}}{f_{k,n}} \longrightarrow 0 \quad \textit{locally uniformly on } \Omega, \quad \textit{as} \quad n \to \infty. \tag{1.2}$$

Such an index k is called a dominant index *of the C-class.*

The dominant index is fixed (independent of n). Every C-class has cardinality at least two, since for a singleton $I = \{k\}$ the normalized sum in (1.2) is identically equal to 1.

In [Car28, pp. 312–318], Cartan proved:

Cartan's theorem. *Under the assumption* (1.1)*, after selection of a subsequence, either all indices form a single C-class or there are at least two disjoint C-classes on $\mathbb{D}$.*

In particular, a partition into C-classes exists on the whole disk for $p \leq 4$. His conjecture that such a partition exists for every $p \geq 3$ was disproved in [Ere96a], where counterexamples were constructed for all $p \geq 5$. Then the following modification was proposed in [Ere96b, Ere15]:

Modified Cartan conjecture. *After passage to a subsequence, every system* (1.1) *admits a partition into C-classes on a concentric disk of positive radius depending only on p.*

This assertion was proved for $p = 5$ in [Ere96b]. More recently, Yamanoi [Yam26, Theorem 1.7] obtained, after passage to a subsequence, disjoint nonempty C-classes $I_1, \dots, I_q$ on the whole disk, together with control of the remaining indices away from exceptional sets of arbitrarily small total radius. More precisely, set $I = I_1 \sqcup \cdots \sqcup I_q$. Then, for every $j \notin I$, every $0 < s < 1$, every $\gamma > 0$, and every $\varepsilon > 0$, one has, for all sufficiently large n,

$$\frac{|f_{j,n}(z)|}{\left(\sum_{k \in I} |f_{k,n}(z)|^2\right)^{1/2}} < \varepsilon$$

for γ-almost all $z \in D(s)$, where "γ-almost all" means outside a set covered by at most countably many closed disks whose sum of radii is less than γ.

This result does not imply that every index belongs to one of the C-classes. Our main theorem establishes the full partition of $\{1, \dots, p\}$ into C-classes after a restriction to a fixed disk, and proves the Modified Conjecture.

Theorem 1.2 (Partition theorem)**.** *For every integer $p \geq 3$ there is $\varepsilon_p \in (0, 1)$ such that every sequence of p-tuples of units on $\mathbb{D}$ satisfying* (1.1) *has a subsequence for which $\{1, \dots, p\}$ is a disjoint union of C-classes on $D(\varepsilon_p)$.*

The statement is evidently conformally invariant: it holds not only for the disk $D(\varepsilon_p)$ but for every hyperbolic disk of the same hyperbolic radius.

Certainly the whole unit disk can be covered by hyperbolic disks of fixed radius but this does not imply that the conclusion of the theorem holds in the whole unit disk: the reason is that a partition into C-classes for a given sequence of p-tuples is not unique, so one can have different partitions on overlapping disks.

For $p = 5$ we obtain the optimal value of ε_p, and can replace disks by arbitrary regions of fixed hyperbolic diameter. We use the hyperbolic length element $2|dz|/(1 - |z|^2)$, denote the hyperbolic distance by d, and define $\operatorname{diam}_{\mathbb{D}} \Omega = \sup_{z,w\in\Omega} d(z, w)$.

Theorem 1.3. *Let $\Omega \subset \mathbb{D}$ be a non-empty open set with $\operatorname{diam}_{\mathbb{D}} \Omega \leq \log 3$. Every sequence of five-tuples of units on $\mathbb{D}$ whose sum is zero has a subsequence admitting a partition into C-classes on Ω.*

The constant $\log 3$ cannot be increased. Let R_p denote the least upper bound of radii ε of Euclidean disks centered at the origin for which the conclusion of Theorem 1.2 holds with $\varepsilon_p = \varepsilon$. Then

$$R_5 = 2 - \sqrt{3}.$$

Indeed, the hyperbolic diameter of $D(R)$ is $4 \operatorname{arctanh} R$; when $R = 2 - \sqrt{3}$ we obtain the hyperbolic diameter $\log 3$.

A p-tuple of units satisfying (1.1) defines a holomorphic curve in the hyperplane $\sum_{j=1}^{p} x_j = 0$ of $\mathbb{P}^{p-1}$, which is isomorphic to $\mathbb{P}^{p-2}$. The curve omits the p coordinate hyperplanes restricted to this hyperplane; these are in general position.

The connection between geometry of hyperplane complements and holomorphic curves in them was studied by many authors beginning with Kiernan and Kobayashi [KK73]; Lang [Lan87, Chapter VIII] gives a systematic account. Our Theorem 1.2 also has an application to the Kobayashi–Royden pseudometric on arbitrary smooth closed algebraic subvarieties of the algebraic torus $(\mathbb{C}^*)^N$ (that is, a product of copies of $\mathbb{C}^*$). For a complex manifold Y, we use the normalization

$$\kappa_Y(x; v) = \inf\{t^{-1} : t > 0,\ F \in \mathcal{O}(\mathbb{D}, Y),\ F(0) = x,\ F'(0) = tv\}$$

of the Kobayashi–Royden pseudometric [Roy71]. Here $\mathcal{O}(\mathbb{D}, Y)$ denotes the set of holomorphic maps from $\mathbb{D}$ to Y. Yamanoi [Yam26, Theorem 1.5] proved that a zero direction for this pseudometric is realized by an entire curve when Y is a smooth closed subvariety of an abelian variety. He [Yam26, p. 459 and Remark 14.4 on p. 604] notes that it is unknown whether the same conclusion holds for noncompact semi-abelian varieties. The following theorem answers Yamanoi's question for smooth closed algebraic subvarieties of the algebraic torus and specifies the entire curve.

For $x = (x_1, \dots, x_N) \in (\mathbb{C}^*)^N$ and $v = (v_1, \dots, v_N) \in T_x(\mathbb{C}^*)^N$, put

$$\lambda_j = \frac{v_j}{x_j}, \qquad \gamma_{x,v}(\zeta) = (x_1 e^{\lambda_1 \zeta}, \dots, x_N e^{\lambda_N \zeta}).$$

Theorem 1.4. *Let $Y \subset (\mathbb{C}^*)^N$ be a smooth closed algebraic subvariety. For $x \in Y$ and $v \in T_x Y$,*

$$\kappa_Y(x; v) = 0 \quad \Longleftrightarrow \quad \gamma_{x,v}(\mathbb{C}) \subset Y.$$

The zero directions form a closed algebraic subset of TY, and κ_Y has a positive lower bound on every compact subset of its complement. These additional properties are proved in Section 6. For the complement of p hyperplanes in $\mathbb{P}^{p-2}$ associated with (1.1), the theorem gives an explicit description in terms of independently rescaled groups of coordinates whose sums are zero.

The main analytic step in the proof of Theorem 1.2 is the following estimate (Proposition 3.1).

Let $A_{i,n}$ be units and $a_{i,n}$ holomorphic functions on $\mathbb{D}$, where $1 \le i \le m$. Suppose that

$$\frac{a_{i,n}}{A_{i,n}} \longrightarrow 0 \quad (1 \le i \le m), \qquad s_n := \sum_{i=1}^{m} a_{i,n} \longrightarrow s \not\equiv 0$$

locally uniformly on $\mathbb{D}$. Then there exist an index i and a subsequence such that $1/A_{i,n} \longrightarrow 0$ locally uniformly on $D(r_m)$, where $r_m > 0$ depends only on m.

The functions $a_{i,n}$, their sums s_n, and the limit s may have zeros. We prove the estimate by induction, using a quantitative estimate for Wronskians and logarithmic derivatives.

The quantitative estimate of the Wronskian is another new ingredient which may have independent interest (Proposition 2.2). It shows stability of the classical theorem that holomorphic functions are linearly dependent iff their Wronskian is identically equal to zero. We show that if the Wronskian is small, and the functions holomorphic in $\mathbb{D}$ are properly normalized, then the functions are almost linearly dependent, in the sense that there is a linear combination, whose coefficients have unit ℓ^2-norm, which is small (in a smaller disk).

To derive this, we use the tools employed by Cartan: the Bloch–Cartan estimate (Lemma 2.1), the lemma on the logarithmic derivative (Lemma 2.3), the Borel–Nevanlinna growth lemma (Lemma 2.4), Harnack's inequality (Lemma 2.5), and Jensen's inequality (Lemma 2.6).

We mention that there is an alternative method of proving Proposition 2.2 based on the recent Universal Plücker-coordinate theorem of Karp and Purbhoo [KP26], but we choose an exposition in the classical spirit.

Lang notes in [Lan87, Chapter VIII, p. 225] that he was unable to simplify Cartan's arguments. Theorem 1.2 does not imply Cartan's result, because its conclusion only applies to a smaller disk.

In Section 2 we establish the analytic estimates. In Section 3 we prove Proposition 3.1, and in Section 4 deduce Theorem 1.2. Section 5 contains the sharp five-function result, and Section 6 treats the geometric applications.

Throughout the paper,

$$\|h\|_r = \max_{|z|\le r} |h(z)|, \qquad \langle u \rangle_r = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta, \qquad m(r,h) = \left\langle \log^+ |h| \right\rangle_r,$$

where $\log^+ t = \max\{0, \log t\}$ and $\log^- t = \max\{0, -\log t\}$.

We use $\|f\|_{L^p}$ for the L^p-norm of a scalar-valued function f, with the domain and measure specified when used. For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$, we use $\|\boldsymbol{a}\|_{\ell^p}$ to denote its ℓ^p-norm; in particular, $\|\boldsymbol{a}\|_{\ell^2}$ is its Euclidean norm.

All limits of holomorphic functions are locally uniform unless stated otherwise. Subsequences are relabelled by n. Constants denoted by C may change from line to line, with their fixed parameters indicated when needed.

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2. Quantitative analytic estimates

Wronskians.

For holomorphic functions $g_1, \dots, g_m$, write

$$W(g_1, \dots, g_m) = \det(g_j^{(i-1)})_{i,j=1}^m.$$

The Wronskian vanishes identically precisely when the functions are linearly dependent over the constants; see [BD10, Theorem 1]. The main result of this section is a quantitative estimate of the Wronskian from below.

We first state a lower estimate of Bloch–Cartan type; compare Cartan's lemma in [Car28, Théorème III, pp. 273–276], or [Lan87, Chap. VIII, §3]. The proof below gives the precise power dependence needed here.

Lemma 2.1. *Fix $0 < a < b < c < 1$. There is $\gamma > 0$, depending only on a, b, c, such that every $F \in \mathcal{O}(\mathbb{D})$ with $|F| \leq 1$ and $\|F\|_a \geq t$, where $0 < t \leq 1$, admits a radius $\rho \in (b, c)$ for which*

$$\min_{|z|=\rho} |F(z)| \geq t^\gamma. \tag{2.1}$$

Proof. If $t = 1$, the maximum principle makes F a constant of modulus one. Suppose that $t < 1$. Choose fixed radii $c < T_- < T_+ < S < 1$, and choose $T \in (T_-, T_+)$ so that F has no zero on $|z| = T$. List its zeros in $D(T)$, with multiplicities, as $a_1, \dots, a_N$. For $|\zeta| < S$ put

$$b_{\zeta,S}(z) = \frac{S(z - \zeta)}{S^2 - \overline{\zeta} z}.$$

Choose $w \in \overline{D(a)}$ with $|F(w)| \geq t$. There is a fixed $q < 1$ such that $|b_{\zeta,S}(w)| \leq q$ whenever $|\zeta| \leq T_+$. Dividing F by the corresponding N factors and applying the maximum principle on $D(S)$ gives $t \leq q^N$, hence

$$N \leq C \log(1/t). \tag{2.2}$$

In $D(T)$ factorize $F = B_T Q$, where $B_T = \prod_{\nu=1}^N b_{a_\nu,T}$. The function Q is zero-free, $|Q| \leq 1$, and $|Q(w)| \geq t$. Harnack's inequality for $-\log|Q|$ gives

$$\log|Q(z)| \geq -C \log(1/t) \qquad (|z| \leq c), \tag{2.3}$$

uniformly for $T \in (T_-, T_+)$.

For $|z| = \rho \in (b, c)$,

$$|b_{a_\nu,T}(z)| \geq \frac{|\rho - |a_\nu||}{2}, \qquad \sup_{0 \leq u \leq T_+} \int_b^c \log \frac{2}{|\rho - u|}\, d\rho < \infty.$$

If $N > 0$, averaging the sum of these logarithms in ρ supplies a radius in (b, c) for which

$$\sum_{\nu=1}^N \log \frac{2}{|\rho - |a_\nu||} \leq CN. \tag{2.4}$$

If $N=0$, any radius in (b,c) suffices. Combining (2.2), (2.3) and (2.4) we prove (2.1). $\square$

For $\boldsymbol{g}=(g_1,\dots,g_m)$ and $0<a<1$, put

$$\Lambda_a(\boldsymbol{g})=\inf\left\{\left\|\sum_{j=1}^m \lambda_j g_j\right\|_a : \boldsymbol{\lambda}\in\mathbb{C}^m, \|\boldsymbol{\lambda}\|_{\ell^2}=1\right\}.$$

The infimum is attained on the compact coefficient sphere. Lemma 2.1 permits an elementary induction on the number of functions.

Proposition 2.2. *For every $m\ge 1$ and $0<a<b<1$, there are constants $c>0$ and $K\ge m$, depending only on m,a,b, such that holomorphic functions $g_1,\dots,g_m$ with $|g_j|\le 1$ on $\mathbb{D}$ satisfy*

$$\|W(g_1,\dots,g_m)\|_b \ge c\,\Lambda_a(g_1,\dots,g_m)^K. \tag{2.5}$$

Proof. Put $\tau=\Lambda_a(\boldsymbol{g})$, so $0\le\tau\le 1$. The case $\tau=0$ is immediate, and for $m=1$ take $c=K=1$. Assume the result for $m-1$, and choose

$$a<b_1<b_2<b_3<b.$$

Set $V=W(g_1,\dots,g_{m-1})$ and $\Delta=\|W(g_1,\dots,g_m)\|_b$. The infimum in Λ_a for the first $m-1$ functions is at least τ. The induction hypothesis therefore gives $\|V\|_{b_1}\ge c_1\tau^{K_1}$. Cauchy's estimates bound V on every fixed smaller disk. Normalize V by such a bound on a disk of radius between b_3 and 1, and rescale that disk to $\mathbb{D}$. Lemma 2.1 then gives $\rho\in(b_2,b_3)$ and constants $c_2,\beta>0$, depending only on the fixed data, such that

$$\min_{|z|=\rho}|V(z)|\ge c_2\tau^{\beta}. \tag{2.6}$$

Put $q=m-1$. On a neighborhood of the circle $|z|=\rho$, we define

$$Y=(g_j^{(i-1)})_{i,j=1}^q, \qquad \boldsymbol{v}=(g_m,g_m',\dots,g_m^{(q-1)})^{\mathsf{T}}, \qquad \boldsymbol{d}=Y^{-1}\boldsymbol{v}.$$

These are single-valued holomorphic functions there, and $\det Y=V$. Differentiating $Y\boldsymbol{d}=\boldsymbol{v}$ shows that the first $q-1$ entries of $Y\boldsymbol{d}'$ vanish; the last is the Schur complement in the full Wronskian. Thus, with $\boldsymbol{e}_q$ the last coordinate vector,

$$\boldsymbol{d}'=Y^{-1}\boldsymbol{e}_q\,\frac{W(g_1,\dots,g_m)}{V}. \tag{2.7}$$

The entries and cofactors of Y are uniformly bounded by Cauchy's estimates. Equations (2.6) and (2.7) imply

$$\|\boldsymbol{d}'(z)\|_{\ell^2}\le C\Delta\tau^{-2\beta} \qquad (|z|=\rho).$$

Fix z_*, $|z_*|=\rho$, and integrate along circular arcs of length at most $2\pi\rho$. The first row of $Y\boldsymbol{d}=\boldsymbol{v}$ then shows that the holomorphic function

$$G(z)=g_m(z)-\sum_{j<m}d_j(z_*)g_j(z)$$

satisfies $\max_{|z|=\rho}|G(z)|\le C\Delta\tau^{-2\beta}$. Its constant coefficient vector has norm at least one. The definition of τ and the maximum principle give

$$\tau\le\|G\|_a\le C\Delta\tau^{-2\beta}.$$

This proves (2.5) with exponent $2\beta+1$. Since $\tau\le 1$, increasing the exponent preserves the inequality, so we may arrange $K\ge m$. $\square$

For the rest of the paper, fix $K_1 = 1$ and, for each $m \geq 2$, an exponent $K_m \geq m$ furnished by Proposition 2.2 for $a = 1/4$ and $b = 1/2$; denote the corresponding constant by c_m. Only the existence of these finite exponents is needed.

Logarithmic derivatives.

We need a localized higher-order form of the classical lemma on the logarithmic derivative, with an interior maximum in place of a value at the origin; compare [Lan87, Chapter VIII]. For sharper classical estimates, see Miles [Mil92]. We give the required version.

Lemma 2.3. *Fix $0 < \alpha < r_- < r_+ < 1$ and an integer $k \geq 1$. There is a constant C, depending only on these parameters, such that every $h \in \mathcal{O}(\mathbb{D})$ with $\|h\|_\alpha \geq \tau > 0$ satisfies*

$$m\left(r, \frac{h^{(k)}}{h}\right) \leq C\left\{\log\left(2 + m(R,h) + \log^+\frac{1}{\tau}\right) + \log\frac{1}{R-r}\right\} \qquad (r_- \leq r < R \leq r_+). \tag{2.8}$$

Proof. All constants may depend on the fixed radii and on k. Put $S = (r+R)/2$ and $d = S - r$. Poisson comparison gives

$$\log|h(z)| \leq H \quad (|z| \leq S), \qquad H = 1 + \frac{C\, m(R,h)}{R-r}.$$

Write $E = H + \log^+(1/\tau)$, so $E \geq 1$. Choose $r + d/3 < T < r + 2d/3$ such that h has no zeros on $|z| = T$. Let $a_1, \ldots, a_N$ be the zeros of h in $D(T)$, counted with multiplicities, and choose $w \in \overline{D(\alpha)}$ with $|h(w)| \geq \tau$.

For $|a| < S$, define the Blaschke factor

$$b_{a,S}(z) = \frac{S(z-a)}{S^2 - \overline{a}z}.$$

It satisfies

$$1 - |b_{a,S}(w)|^2 = \frac{(S^2 - |a|^2)(S^2 - |w|^2)}{|S^2 - \overline{a}w|^2}.$$

Since $|a| \leq T$, $S - T \geq d/3$, and $|w| \leq \alpha < r_-$, we obtain $|b_{a,S}(w)| \leq e^{-cd}$ with a uniform $c > 0$. Divide h by the product of these factors over its zeros in $D(T)$. The quotient is holomorphic on $D(S)$ and bounded by e^H on its boundary, so the maximum principle gives $\tau \leq e^H e^{-cdN}$. Thus

$$N \leq CE/d. \tag{2.9}$$

Now factor $h = B_T q$ in $D(T)$, where $B_T = \prod_{\nu=1}^N b_{a_\nu,T}$. The function q is zero-free, $|q| \leq e^H$, and $|q(w)| \geq \tau$. Harnack's inequality for $v = H - \log|q|$ gives $v(0) \leq CE$. Choose a holomorphic logarithm of q on $D(T)$ and apply the Schwarz integral formula to $H - \log q$ on the circle of radius $(r+T)/2$. The integral of its real part is $v(0)$, while its imaginary constant disappears upon differentiation. We obtain

$$|(\log q)^{(j)}(z)| \leq C_j E/d^{j+1} \qquad (|z| = r,\ 1 \leq j \leq k). \tag{2.10}$$

Define the meromorphic functions $L_1(h) = h'/h$ and $L_{j+1}(h) = L_j(h)'$. Differentiating the Blaschke factors and using (2.9)–(2.10) gives

$$|L_j(h)(z)| \leq \frac{C_j E}{d^{j+1}} + C_j \sum_{\nu=1}^N |z - a_\nu|^{-j} \qquad (|z| = r). \tag{2.11}$$

Indeed, the reflected poles are at distance at least $T - r$ from this circle and contribute at most $C_j N/d^j$. The estimate is needed only away from any poles on the circle.

Set $\gamma = 1/(2k)$. For $1 \le j \le k$,

$$\sup_{a\in\mathbb{C}} \frac{1}{2\pi}\int_0^{2\pi} |re^{i\theta}-a|^{-j\gamma}\,d\theta \le C_k \qquad (r_- \le r \le r_+). \tag{2.12}$$

If $|a| \notin [r/2, 3r/2]$, the integrand is uniformly bounded. Otherwise rotate a to the positive real axis and use

$$|re^{i\theta}-a|^2 = (r-|a|)^2 + 2r|a|(1-\cos\theta) \ge c\theta^2 \qquad (-\pi \le \theta \le \pi),$$

with $j\gamma \le 1/2 < 1$.

Raise (2.11) to the power γ, use $(x+y)^\gamma \le x^\gamma + y^\gamma$, and integrate. By (2.9), (2.12), and concavity of the logarithm,

$$\begin{aligned} m(r, L_j(h)) &\le \frac{1}{\gamma}\log\Big(1 + \langle |L_j(h)|^\gamma\rangle_r\Big) \\ &\le C_k\left(1 + \log E + \log\frac{1}{d}\right). \end{aligned}$$

Repeated differentiation shows that $h^{(k)}/h$ is a polynomial with fixed coefficients in $L_1(h), \dots, L_k(h)$. The same bound holds for $m(r, h^{(k)}/h)$. Substituting E, H, and d proves (2.8). □

Three auxiliary estimates.

We first state an elementary form of the Borel–Nevanlinna growth lemma; compare [Lan87, Chap. VI, Lemma 3.7].

Lemma 2.4. *Let M be positive, continuous, and nondecreasing on $[a,b]$, where $a<b$ and $2/M(a) < b-a$. There is $\rho \in [a,b)$ such that*

$$\rho + \frac{1}{M(\rho)} < b, \qquad M\left(\rho + \frac{1}{M(\rho)}\right) \le 2M(\rho). \tag{2.13}$$

Proof. Start with $t_0 = a$. If the second inequality in (2.13) fails at t_j, put $t_{j+1} = t_j + 1/M(t_j)$. After j failures, $M(t_j) \ge 2^j M(a)$, so all proposed points lie below $a + 2/M(a) < b$. An infinite sequence would converge to a point of $[a,b)$ while $M(t_j)$ tends to infinity, contradicting continuity. The first stopping point has both required properties. □

Lemma 2.5. *Suppose that $1/2 \le \rho \le 3/4$ and $0 < \delta \le \eta \le 1/64$. Let $u_1, \dots, u_m$ be real harmonic functions on a neighborhood of $\overline{D(\rho)}$, with $\max_i u_i > 0$ on $|z| = \rho$. Let U be the harmonic extension of this boundary maximum and put $M = U(0)$. If, for each i, there is $z_i \in D(\delta)$ such that $u_i(z_i) \le C_0$, where $C_0 \ge 0$, then*

$$u_i(0) \le 8\delta M + C_0, \tag{2.14}$$

$$\sup_{|z|\le 4\eta} u_i(z) \le 64\eta M + C_0. \tag{2.15}$$

Proof. The functions U and $U - u_i$ are nonnegative harmonic functions. Put $h = (\rho-\delta)/(\rho+\delta)$. Harnack's inequality applied to these two functions gives

$$(U-u_i)(0) \ge h(U-u_i)(z_i) \ge h^2 M - C_0.$$

Since $1 - h^2 \le 8\delta$, this proves (2.14). For $|z| \le 4\eta$, put $h_4 = (\rho - 4\eta)/(\rho + 4\eta)$ and apply Harnack once more:

$$u_i(z) \le (h_4^{-1} - h_4 h^2)M + C_0.$$

The coefficient satisfies

$$h_4^{-1} - h_4 h^2 \le (h_4^{-1} - h_4) + (1 - h^2) \le \frac{32\eta}{1 - 64\eta^2} + 8\eta \le 64\eta,$$

which proves (2.15). □

The next estimate is an off-center version of Jensen's inequality for $\log|H|$. When $w = 0$, it reduces to $\log|H(0)| \le \langle \log|H| \rangle_\rho$; the Poisson kernel bounds below quantify the loss when $w \ne 0$.

Lemma 2.6. *Let H be holomorphic on a neighborhood of $\overline{D(\rho)}$, and let $w \in D(\rho)$ with $H(w) \ne 0$. Put $t = |w|/\rho$ and $q = (1+t)/(1-t)$. Then*

$$\langle \log|H| \rangle_\rho \ge q \log|H(w)| - (q^2 - 1) m(\rho, H). \tag{2.16}$$

Proof. Let P be the Poisson kernel at w, normalized for angular measure $d\theta/(2\pi)$, so $q^{-1} \le P \le q$. Writing $I_+ = m(\rho, H)$ and $I_- = \left\langle \log^- |H| \right\rangle_\rho$, subharmonic comparison gives

$$\log|H(w)| \le qI_+ - q^{-1}I_- = q^{-1}(I_+ - I_-) + (q - q^{-1})I_+.$$

Multiplication by q proves (2.16). At boundary zeros, apply the comparison to $\log(|H| + \varepsilon)$ and let $\varepsilon \downarrow 0$; the logarithmic singularities are integrable. □

3. Growth estimate for units

With the exponents K_m fixed in Proposition 2.2, put

$$r_1 = 1, \qquad r_2 = 2 - \sqrt{3}, \qquad \eta_m = \frac{1}{1024(K_m + m)} \ (m \ge 2), \qquad r_m = \eta_m r_{m-1} \quad (m \ge 3). \tag{3.1}$$

The radii r_m are positive and decrease with m. It is convenient to use the sharp value of r_m at $m = 2$ which is proved in Proposition 5.3 below; its proof uses only the auxiliary estimates of Section 2, and is independent of the contents of this Section 3.

Proposition 3.1. *Let $m \ge 1$. For every n, let $A_{1,n}, \ldots, A_{m,n}$ be units on $\mathbb{D}$, and let $a_{1,n}, \ldots, a_{m,n} \in \mathcal{O}(\mathbb{D})$. Suppose that*

$$\frac{a_{i,n}}{A_{i,n}} \longrightarrow 0 \quad (1 \le i \le m), \qquad s_n := \sum_{i=1}^{m} a_{i,n} \longrightarrow s \not\equiv 0 \tag{3.2}$$

locally uniformly on $\mathbb{D}$. There exist an index i and a subsequence such that $1/A_{i,n} \to 0$ locally uniformly on $D(r_m)$, where r_m is defined by (3.1).

Proof. We argue by induction on m.

For $m = 1$, let $K \Subset \mathbb{D}$ be arbitrary. Choose a concentric circle containing K and avoiding the zeros of s. Since $a_{1,n} = s_n \to s$ locally uniformly, $a_{1,n}$ is bounded away from zero on this circle for all sufficiently large n. Hence

$$\frac{1}{A_{1,n}} = \frac{a_{1,n}/A_{1,n}}{a_{1,n}} \longrightarrow 0, \quad n \to \infty$$

uniformly on the circle. The maximum principle for the holomorphic function $1/A_{1,n}$ then gives convergence uniformly on the enclosed disk. Since K was arbitrary, the assertion follows for $r_1 = 1$.

For $m = 2$, the conclusion follows from Proposition 5.3, and therefore in particular holds on $D(r_2)$.

Let $m \geq 3$, assume the result for $m-1$, and abbreviate

$$\eta = \eta_m, \qquad \delta = r_m = \eta r_{m-1}.$$

Assume for contradiction that there is no fixed index i and no subsequence for which $1/A_{i,n} \to 0$ locally uniformly on $D(\delta)$. Every further extraction below preserves this assumption.

A lower bound for linear combinations. For each i, zero is not a subsequential limit of $1/A_{i,n}$ in the compact-open topology on $D(\delta)$. Hence there exist a compact set $K_i \Subset D(\delta)$ and a number $\varepsilon_i > 0$ such that

$$\sup_{K_i} |1/A_{i,n}| \geq \varepsilon_i$$

for all sufficiently large n. Otherwise, a diagonal selection along an exhaustion of $D(\delta)$ by compact disks would produce the forbidden convergence to zero. Then we may choose a common constant $C_0 \geq 0$ and points $z_{i,n} \in D(\delta)$ such that

$$u_{i,n}(z_{i,n}) \leq C_0, \qquad u_{i,n} = \log |A_{i,n}|, \quad 1 \leq i \leq m. \tag{3.3}$$

The functions $u_{i,n}$ are harmonic because the $A_{i,n}$ are units.

Consider the least norm of a linear combination whose coefficient vector has Euclidean norm one,

$$\lambda_n = \Lambda_\eta(a_{1,n}, \dots, a_{m,n}).$$

We claim that there is a constant $\lambda > 0$, independent of n, such that

$$\lambda_n \geq \lambda \tag{3.4}$$

for all sufficiently large n.

If not, pass to a subsequence for which $\lambda_n \to 0$ and choose minimizing unit vectors $\boldsymbol{c}_n = (c_{1,n}, \dots, c_{m,n}) \in \mathbb{C}^m$. Put

$$b_n = \sum_i c_{i,n} a_{i,n}.$$

After a further extraction of a subsequence, one fixed index j satisfies

$$|c_{j,n}| = \max_i |c_{i,n}| \geq m^{-1/2}$$

for every n. Since $\|b_n\|_\eta = \lambda_n \to 0$, the functions b_n tend uniformly to zero on $\overline{D(\eta)}$, and

$$\sum_{i \neq j} \left(1 - \frac{c_{i,n}}{c_{j,n}}\right) a_{i,n} = s_n - \frac{b_n}{c_{j,n}} \longrightarrow s$$

locally uniformly on $D(\eta)$. The coefficients on the left have modulus at most two. Retaining the units $A_{i,n}$ for $i \neq j$ and rescaling $D(\eta)$ to $\mathbb{D}$, the induction hypothesis applies to these $m-1$ terms. It follows that, for some fixed $i \neq j$ and a subsequence,

$$\frac{1}{A_{i,n}} \longrightarrow 0 \qquad \text{locally uniformly on } D(\eta r_{m-1}) = D(\delta),$$

contrary to our standing assumption. This proves (3.4). In particular,

$$\|a_{i,n}\|_\eta \geq \lambda \qquad (1 \leq i \leq m).$$

A circle with controlled growth. Discard finitely many terms so that $|a_{i,n}| \le |A_{i,n}|$ on $\overline{D(7/8)}$ for every i. For $0 < t < 1$, define

$$M_n(t) = \langle \max\{0, u_{1,n}, \dots, u_{m,n}\} \rangle_t .$$

This function is continuous and nondecreasing in t, since it is the circular mean of a subharmonic function. Moreover,

$$M_n(1/2) \longrightarrow \infty. \tag{3.5}$$

Indeed, choose $z_* \in D(1/2)$ with $s(z_*) \neq 0$. From (3.2),

$$|s_n(z_*)| \le \sum_i \left| \frac{a_{i,n}}{A_{i,n}}(z_*) \right| |A_{i,n}(z_*)|.$$

Since $s_n(z_*) \to s(z_*) \neq 0$, whereas $a_{i,n}/A_{i,n} \to 0$, we obtain $\max_i |A_{i,n}(z_*)| \to \infty$. Poisson comparison for the nonnegative subharmonic function $\max\{0, u_{1,n}, \dots, u_{m,n}\}$ then proves (3.5).

Choose a fixed closed interval $[a, b] \subset (1/2, 3/4)$ such that s has no zeros in the closed annulus $a \le |z| \le b$. Such an interval exists because $s \not\equiv 0$. Hence $|s_n|$ is bounded below by a positive constant on this annulus for all sufficiently large n. By (3.5) and Lemma 2.4, we may choose $\rho = \rho_n \in [a, b]$ and put

$$M = M_n(\rho), \qquad \widehat{\rho} = \rho + M^{-1} < b, \qquad M_n(\widehat{\rho}) \le 2M. \tag{3.6}$$

Here and below $M \to \infty$ as $n \to \infty$.

On $|z| = \rho$, the positive lower bound for $|s_n|$ and the uniform smallness of all $a_{i,n}/A_{i,n}$ imply $\max_i u_{i,n} > 0$ for all sufficiently large n. Let U_n be the harmonic extension of this boundary maximum. Then

$$U_n > 0, \qquad U_n(0) = M, \qquad U_n - u_{i,n} \ge 0.$$

Applying Lemma 2.5 with (3.3), we obtain

$$u_{i,n}(0) \le 8\delta M + C_0 \tag{3.7}$$

and

$$\|a_{i,n}\|_{4\eta} \le e^{L_n}, \qquad L_n = 64\eta M + C_0. \tag{3.8}$$

The Wronskian lower bound. Normalize on the small disk by setting

$$g_{i,n}(z) = e^{-L_n} a_{i,n}(4\eta z), \qquad z \in \mathbb{D}.$$

By (3.8), these functions have modulus at most one. By (3.4), the minimum norm of a linear combination with coefficient vector of Euclidean norm one on $D(1/4)$ is at least $e^{-L_n}\lambda$. Put $W_n = W(a_{1,n}, \dots, a_{m,n})$. The exact scaling identity is

$$W(g_{1,n}, \dots, g_{m,n})(z) = e^{-mL_n}(4\eta)^{m(m-1)/2} W_n(4\eta z).$$

Proposition 2.2 therefore gives

$$\|W_n\|_{2\eta} \ge c_m (4\eta)^{-m(m-1)/2} \lambda^{K_m} e^{-(K_m - m)L_n}.$$

Choose $w_n \in \overline{D(2\eta)}$ at which $|W_n|$ attains this norm. Since $K_m \ge m$ and $L_n = 64\eta M + C_0$, we obtain

$$\log |W_n(w_n)| \ge -64 K_m \eta M - O(1). \tag{3.9}$$

The $O(1)$ term may depend on the fixed data of the sequence, through λ and C_0, but it is independent of n. In particular, $W_n \not\equiv 0$.

The boundary estimate. Apply Lemma 2.3 with interior radius η, fixed outer range $r_- = 1/2$, $r_+ = 7/8$, lower bound λ, and the radii $\rho, \widehat{\rho}$ from (3.6). Since $m(\widehat{\rho}, a_{i,n}) \le M_n(\widehat{\rho}) \le 2M$ and $\widehat{\rho} - \rho = M^{-1}$, we have

$$m\left(\rho, \frac{a_{i,n}^{(k)}}{a_{i,n}}\right) = O(\log M) = o(M) \qquad (1 \le i \le m,\ 1 \le k \le m-1). \tag{3.10}$$

For each j, multilinearity and $s_n = \sum_i a_{i,n}$ give

$$W_n = \pm W(s_n, a_{1,n}, \ldots, \widehat{a_{j,n}}, \ldots, a_{m,n}),$$

where the hat denotes omission. Since $s_n \to s$ locally uniformly, the derivatives of s_n through order $m-1$ are uniformly bounded on $\overline{D(3/4)}$. Factoring the remaining columns and expanding the determinant, we obtain a nonnegative function E_n on $|z| = \rho$, independent of j, such that, almost everywhere,

$$\log|W_n| \le \sum_{i \ne j} u_{i,n} + E_n, \qquad \langle E_n \rangle_\rho = o(M). \tag{3.11}$$

For instance, one may take a sufficiently large fixed constant plus

$$\sum_i \sum_{k=1}^{m-1} \log^+ \left| \frac{a_{i,n}^{(k)}}{a_{i,n}} \right|,$$

in view of (3.10).

At each boundary point choose j so that $u_{j,n}$ is maximal, and integrate. Harmonicity gives

$$I_n := \langle \log|W_n| \rangle_\rho \le \sum_i u_{i,n}(0) - M + o(M). \tag{3.12}$$

For the positive part, put $v_n = \max_i u_{i,n}$ on $|z| = \rho$. Since $v_n > 0$ and $E_n \ge 0$, (3.11) implies

$$\log^+|W_n| \le (m-1)v_n + E_n. \tag{3.13}$$

almost everywhere on this circle. Averaging gives

$$m(\rho, W_n) \le (m-1)M + o(M). \tag{3.14}$$

The logarithmic singularities in (3.11) and (3.13) are integrable, so the means in (3.12) and (3.14) are well-defined.

Apply Lemma 2.6 at w_n. With $t = |w_n|/\rho \le 4\eta$ and $q = (1+t)/(1-t)$, the bound $\eta \le 1/64$ implies

$$q \le 2, \qquad q^2 - 1 \le 20\eta.$$

Combining (3.9), (3.14), and (2.16), we obtain

$$I_n \ge -(128K_m + 20m)\eta M - o(M).$$

On the other hand, (3.7) and (3.12) give

$$I_n \le (8m\delta - 1)M + o(M).$$

Comparing the last two inequalities and using (3.1), we obtain

$$1 \le 8m\delta + (128K_m + 20m)\eta + o(1) \le \frac{128K_m + 28m}{1024(K_m + m)} + o(1) \le \frac{1}{8} + o(1),$$

since $\delta \le \eta$. This is impossible as $n \to \infty$ and completes the induction. $\square$

Reduction under linear dependence.

Linear relations among the functions $a_{i,n}$ in Proposition 3.1 allow a slight improvement of its conclusion:

Corollary 3.2. *Under the hypotheses of Proposition 3.1, suppose in addition that*

$$\dim_{\mathbb{C}} \operatorname{span}\{a_{1,n}, \ldots, a_{m,n}\} \le d \quad \text{for every } n,$$

where d is an integer with $1 \le d \le m$. Then some reciprocal $1/A_{i,n}$ tends to zero, along a subsequence, locally uniformly on $D(r_d)$, where r_d is defined by (3.1).

Proof. Since $s \not\equiv 0$, the span $V_n = \operatorname{span}_{\mathbb{C}}\{a_{1,n}, \ldots, a_{m,n}\}$ is nonzero for all sufficiently large n. After passage to a subsequence, its dimension is a fixed integer q, where $1 \le q \le d$. Choose any linear isomorphism $T_n : V_n \to \mathbb{C}^q$. Among all q-element subsets $I_n \subset \{1, \ldots, m\}$, choose one maximizing $\left|\det\left(T_n a_{i,n}\right)_{i \in I_n}\right|$, with the columns in increasing order of their indices. This maximum is nonzero, so the functions $a_{i,n}$, $i \in I_n$, form a basis. Write

$$a_{j,n} = \sum_{i \in I_n} c_{ij,n} a_{i,n}.$$

Replacing the column indexed by i by $T_n a_{j,n}$ multiplies the determinant by $c_{ij,n}$. By maximality of the chosen determinant, $|c_{ij,n}| \le 1$ for every $i \in I_n$ and every j. There are finitely many possible sets I_n, so after another extraction we may assume that $I_n = I$ is fixed. Put

$$b_{i,n} = \left(\sum_{j=1}^{m} c_{ij,n}\right) a_{i,n} \qquad (i \in I).$$

The coefficients in parentheses have modulus at most m. Hence

$$\sum_{i \in I} b_{i,n} = s_n, \qquad \frac{b_{i,n}}{A_{i,n}} \longrightarrow 0$$

locally uniformly on $\mathbb{D}$. Apply Proposition 3.1 to these q terms and use $r_q \ge r_d$. □

4. Proof of Theorem 1.2

We derive Theorem 1.2 from Proposition 3.1 by comparing quotients on finitely many concentric disks.

A *preorder* is a reflexive and transitive relation $\preceq$. It defines an equivalence relation by $i \sim j$ if both $i \preceq j$ and $j \preceq i$. An element i is *maximal* if $i \preceq j$ implies $j \preceq i$ for every j; an equivalence class is maximal if its elements are maximal. Two elements are *incomparable* if neither precedes the other.

Lemma 4.1. *Let $0 < \sigma < 1$, put $\rho_k = \sigma^k$ for $0 \le k \le p-1$, and let $f_{1,n}, \ldots, f_{p,n}$ be units on $\mathbb{D}$. After passage to a subsequence, for every i, j, k the quotient $f_{i,n}/f_{j,n}$ either converges locally uniformly on $D(\rho_k)$ to a holomorphic function, or satisfies*

$$\max_K |f_{i,n}/f_{j,n}| \longrightarrow \infty$$

for some fixed compact set $K \Subset D(\rho_k)$. For this subsequence define

$$i \preceq_k j \quad \Longleftrightarrow \quad (f_{i,n}/f_{j,n})_n \text{ is locally uniformly bounded on } D(\rho_k).$$

We say that j dominates *i on $D(\rho_k)$ precisely when $i \preceq_k j$. This relation is a preorder. Write $i \sim_k j$ when both $i \preceq_k j$ and $j \preceq_k i$. Then there is $\ell \in \{0, \ldots, p-1\}$ such that either $\preceq_\ell$ has* *just one maximal equivalence class, or $\ell < p-1$ and indices chosen from distinct maximal equivalence classes for $\preceq_\ell$ are incomparable for $\preceq_{\ell+1}$.*

Proof. For each of the finitely many pairs (i,j) and integers k, consider the current sequence of quotients on $D(\rho_k)$. If it is locally uniformly bounded, Montel's theorem [Ahl79, Ch. V, Theorem 15] gives a subsequence converging locally uniformly to a holomorphic function. If it is not locally uniformly bounded, there is a fixed compact subset on which the suprema are unbounded; choose a subsequence along which these suprema tend to infinity. Both conclusions persist under further extraction, so finitely many selections give all the asserted alternatives simultaneously.

Reflexivity of $\preceq_k$ follows from $f_{i,n}/f_{i,n} = 1$, and transitivity follows by multiplying quotients. Restriction to a smaller disk gives

$$i \preceq_k j \quad \Longrightarrow \quad i \preceq_{k+1} j.$$

Let t_k be the number of maximal equivalence classes for $\preceq_k$. Every maximal class for $\preceq_{k+1}$ contains a maximal class for $\preceq_k$. Indeed, given an index x in the former, finiteness of the set of indices gives a maximal index y for $\preceq_k$ with $x \preceq_k y$. Then $x \preceq_{k+1} y$, and maximality of x for $\preceq_{k+1}$ gives $y \preceq_{k+1} x$. The whole $\sim_k$-class of y is therefore contained in the $\sim_{k+1}$-class of x.

Distinct maximal classes for $\preceq_{k+1}$ are disjoint, so they contain distinct maximal classes for $\preceq_k$. Consequently $t_{k+1} \le t_k$. If $t_{k+1} = t_k$, each maximal class for $\preceq_{k+1}$ contains exactly one maximal class for $\preceq_k$, and every maximal class for $\preceq_k$ is contained in one of them. Thus indices belonging to two distinct old maximal classes lie in two distinct new maximal classes. Such indices are incomparable for $\preceq_{k+1}$: a relation in either direction would, by maximality, make the two new classes equal.

If some $t_k = 1$, take $\ell = k$. Otherwise the p nonincreasing integers $t_0, \dots, t_{p-1}$ all belong to $\{2, \dots, p\}$. Hence $t_\ell = t_{\ell+1}$ for some $\ell < p-1$, which gives the second conclusion. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. Set

$$\sigma = r_{p-1}, \qquad \varepsilon_p = \sigma^{p-1}, \qquad \rho_k = \sigma^k \quad (0 \le k \le p-1),$$

and apply Lemma 4.1. For the integer ℓ given by the lemma, choose one index k_ν from each maximal equivalence class for $\preceq_\ell$, with $1 \le \nu \le t$. Every index j satisfies $j \preceq_\ell k_\nu$ for at least one ν. Assign j to one such ν, and let J_ν be the set of indices assigned to ν. If an index belongs to a maximal equivalence class, assign it to the same set as the chosen index k_ν from that class. Then

$$\{1, \dots, p\} = J_1 \sqcup \cdots \sqcup J_t, \qquad k_\nu \in J_\nu, \qquad j \preceq_\ell k_\nu \quad (j \in J_\nu).$$

The sets J_ν are the parts of this partition; they need not be equivalence classes for $\sim_\ell$. Put $F_{\nu,n} = f_{k_\nu,n}$. By the quotient convergence in Lemma 4.1,

$$h_{\nu,n} := \sum_{j \in J_\nu} \frac{f_{j,n}}{F_{\nu,n}} \longrightarrow h_\nu$$

locally uniformly on $D(\rho_\ell)$, where h_ν is holomorphic. Moreover,

$$\sum_{\nu=1}^{t} h_{\nu,n} F_{\nu,n} = 0. \tag{4.1}$$

If $t=1$, the normalized sum is identically zero and the conclusion follows.

Suppose that $t\ge 2$ and that $h_\nu \not\equiv 0$ for some ν. Lemma 4.1 gives $\ell < p-1$ and, for every $\mu\neq\nu$, $k_\nu \not\preceq_{\ell+1} k_\mu$. Thus $(F_{\nu,n}/F_{\mu,n})_n$ is not locally uniformly bounded on $D(\rho_{\ell+1})$. The divergence alternative in that lemma gives a fixed compact set $K_\mu \Subset D(\rho_{\ell+1})$ such that

$$L_{\mu,n} := \max_{K_\mu}\left|\frac{F_{\nu,n}}{F_{\mu,n}}\right| \longrightarrow \infty.$$

Put $B_n = (\min_{\mu\neq\nu} L_{\mu,n})^{1/2}$. Then $B_n\to\infty$ and $L_{\mu,n}/B_n\to\infty$ for every $\mu\neq\nu$. On $D(\rho_\ell)$ define

$$A_{\mu,n} = B_n\frac{F_{\mu,n}}{F_{\nu,n}}, \qquad a_{\mu,n} = h_{\mu,n}\frac{F_{\mu,n}}{F_{\nu,n}} \quad (\mu\neq\nu).$$

These $A_{\mu,n}$ are units, and $a_{\mu,n}/A_{\mu,n} = h_{\mu,n}/B_n \to 0$ locally uniformly. By (4.1),

$$\sum_{\mu\neq\nu} a_{\mu,n} = -h_{\nu,n} \longrightarrow -h_\nu \not\equiv 0.$$

Apply Proposition 3.1 to these $t-1$ terms after rescaling $D(\rho_\ell)$ to $\mathbb{D}$. For some $\mu\neq\nu$, along a further subsequence,

$$\frac{1}{A_{\mu,n}} = \frac{F_{\nu,n}}{B_n F_{\mu,n}} \longrightarrow 0 \quad \text{locally uniformly on } D(\rho_\ell r_{t-1}).$$

Since $r_{t-1}\ge r_{p-1}=\sigma$, this disk contains $D(\rho_{\ell+1})$. This contradicts

$$\max_{K_\mu}\left|\frac{1}{A_{\mu,n}}\right| = \frac{L_{\mu,n}}{B_n} \longrightarrow \infty.$$

Therefore every h_ν vanishes identically. Each J_ν is a C-class on $D(\rho_\ell)$, with dominant index k_ν, and hence also on $D(\varepsilon_p)$. Each contains at least two indices. $\square$

The conclusion is invariant under disk automorphisms.

Corollary 4.2. *For every $z_0\in\mathbb{D}$, the sequence in Theorem 1.2 has, after extraction, a partition into C-classes on*

$$\left\{ z\in\mathbb{D} : \left|\frac{z-z_0}{1-\overline{z_0}z}\right| < \varepsilon_p \right\}.$$

The partition and subsequence may depend on z_0.

Proof. Apply the theorem to the functions composed with $w\mapsto (z_0+w)/(1+\overline{z_0}w)$. $\square$

5. The sharp radius for five functions

Put $r_* = 2-\sqrt{3}$. We use the hyperbolic distance $d_{\mathbb{D}}$ associated with the length element $2|dz|/(1-|z|^2)$. Thus

$$\operatorname{diam}_{\mathbb{D}} D(r) = 4\operatorname{arctanh} r, \qquad 4\operatorname{arctanh} r_* = \log 3. \tag{5.1}$$

For a disk $D(\rho)$, its intrinsic hyperbolic distance will be denoted by $d_{D(\rho)}$.

The harmonic functions used in the example below occur in [Ere96b]. The extremal problem stated in [Ere96b] for the harmonic functions discussed there was solved by Tamrazov [Tam98]; see also the addition of April 24, 1996 in [Ere96b]. We shall instead compare harmonic functions along the geodesic joining two prescribed points.

A comparison along a geodesic.

For real $x \in (-1,1)$ put

$$P_x(\theta) = \frac{1-x^2}{1-2x\cos\theta + x^2}.$$

We first prove an inequality.

Lemma 5.1. *For every $0 < q < r_*$ there is $\varepsilon_q > 0$ such that, for every positive harmonic function U on $\mathbb{D}$ and all real t, s with $0 \le t \le q$ and $|s| \le t$, we have*

$$k_-(t,s)U(-t) + k_+(t,s)U(t) \ge (1+\varepsilon_q)U(s), \tag{5.2}$$

where

$$k_-(t,s) = \frac{(1-t)(1-s)}{(1+t)(1+s)}, \qquad k_+(t,s) = \frac{(1-t)(1+s)}{(1+t)(1-s)}.$$

Proof. By the Poisson representation of a positive harmonic function, it suffices to compare the kernels. Write $c = \cos\theta$. Since $1 + r_*^2 = 4r_*$, we have

$$\frac{1-r_*}{1+r_*}P_{-r_*}(\theta) = \frac{1}{2+c}, \qquad \frac{1-r_*}{1+r_*}P_{r_*}(\theta) = \frac{1}{2-c}.$$

A simple computation gives

$$\begin{aligned} &\frac{1-s}{(1+s)(2+c)} + \frac{1+s}{(1-s)(2-c)} - \frac{1-s^2}{1-2sc+s^2} \\ &\quad = \frac{\big((1+s^2)c - 2s\big)^2 + 12s^2(1-c^2)}{(1-s^2)(4-c^2)(1-2sc+s^2)} \ge 0. \end{aligned} \tag{5.3}$$

For fixed $c \in [-1,1]$,

$$\frac{d}{dt}\frac{(1-t)^2}{1 \mp 2tc + t^2} = -\frac{2(1-t^2)(1 \mp c)}{(1 \mp 2tc + t^2)^2}.$$

Both derivatives are nonpositive, and at least one is negative. Consequently, replacing r_* in the first two kernels by any $t < r_*$ makes the difference in (5.3) strictly positive. After division by $P_s(\theta)$, this difference has a positive minimum on

$$\{(t,s,c) : 0 \le t \le q,\ |s| \le t,\ -1 \le c \le 1\}.$$

Integration against the positive measure representing U proves (5.2). $\square$

The next formulation makes the role of hyperbolic distance explicit.

Corollary 5.2. *Fix $0 < d_0 < \log 3$. There is $\varepsilon > 0$, depending only on d_0, with the following property. Let U, P, Q be nonnegative harmonic functions on $\mathbb{D}$, with $U > 0$, and let $\alpha, \beta \in \mathbb{D}$ satisfy $d_{\mathbb{D}}(\alpha,\beta) \le d_0$. If $C_0 \ge 0$ and*

$$Q(\alpha) \ge U(\alpha) - C_0, \qquad P(\beta) \ge U(\beta) - C_0,$$

then, on the hyperbolic geodesic segment Γ joining α and β we have

$$U(z) - P(z) - Q(z) \le -\varepsilon U(z) + 2C_0. \tag{5.4}$$

Proof. An automorphism of $\mathbb{D}$ takes the endpoints of the segment to $-t, t$, with $4\,\mathrm{arctanh}\, t = d_{\mathbb{D}}(\alpha,\beta)$. Put $q = \tanh(d_0/4) < r_*$. On the resulting real segment, Harnack's inequality gives

$$Q(s) \ge k_-(t,s)Q(-t), \qquad P(s) \ge k_+(t,s)P(t).$$

Both coefficients belong to $[0, 1]$. Adding these inequalities and applying Lemma 5.1 proves (5.4), with $\varepsilon = \varepsilon_q$. Composition with the inverse automorphism returns to Γ. Coincident endpoints are included by taking $t = 0$. □

The estimate for two terms.

The following is the more precise version of Proposition 3.1 for the case $m = 2$.

Proposition 5.3. *Let A_n, B_n be units on $\mathbb{D}$, and let $a_n, b_n \in \mathcal{O}(\mathbb{D})$ satisfy*

$$a_n/A_n \longrightarrow 0, \qquad b_n/B_n \longrightarrow 0, \qquad s_n := a_n + b_n \longrightarrow s \not\equiv 0$$

locally uniformly on $\mathbb{D}$. Let Ω be a nonempty open subset of $\mathbb{D}$ with $\operatorname{diam}_{\mathbb{D}} \Omega \le \log 3$. After passage to a subsequence, either $1/A_n \to 0$ or $1/B_n \to 0$ locally uniformly on Ω.

Proof. Suppose that neither reciprocal tends to 0 on any subsequence. After discarding finitely many terms, there are compact sets $K_A, K_B \Subset \Omega$, a constant $C_0 \ge 0$, and points $\alpha_n \in K_A$, $\beta_n \in K_B$ such that

$$u_n(\alpha_n) \le C_0, \qquad v_n(\beta_n) \le C_0, \qquad u_n = \log |A_n|, \quad v_n = \log |B_n|. \tag{5.5}$$

Put $K = K_A \cup K_B$. Since $K \Subset \Omega$, there is $\eta > 0$ such that the closed hyperbolic η-neighborhood of K is contained in Ω. Hence

$$\operatorname{diam}_{\mathbb{D}} K < \log 3.$$

Indeed, if $x, y \in K$ satisfied $d_{\mathbb{D}}(x, y) = \log 3$, we could extend the hyperbolic geodesic from x through y by a distance smaller than η, obtaining a point of Ω at distance greater than $\log 3$ from x. Choose a neighborhood V of K with compact closure in Ω and

$$\operatorname{diam}_{\mathbb{D}} \overline{V} < \log 3.$$

Surround the finitely many zeros of s in K by pairwise disjoint closed disks contained in V, with no zero of s on their boundaries. If α_n lies inside one of these disks, the maximum modulus principle for $1/A_n$ permits us to move it to the boundary while preserving (5.5). Move β_n in the same way, using $1/B_n$. The resulting points lie in a fixed compact set

$$K' \subset V \setminus \{s = 0\}.$$

Thus $|s_n| \ge c_0 > 0$ on K' for all sufficiently large n. Since $a_n/A_n \to 0$ locally uniformly and $|A_n(\alpha_n)| \le e^{C_0}$ by (5.5), we have $a_n(\alpha_n)/s_n(\alpha_n) \to 0$. Similarly, $b_n(\beta_n)/s_n(\beta_n) \to 0$, and therefore $a_n(\beta_n)/s_n(\beta_n) = 1 - b_n(\beta_n)/s_n(\beta_n) \to 1$. Thus

$$\frac{a_n(\alpha_n)}{s_n(\alpha_n)} \longrightarrow 0, \qquad \frac{a_n(\beta_n)}{s_n(\beta_n)} \longrightarrow 1. \tag{5.6}$$

Choose $r < 1$ with $K' \subset D(r)$. On $K' \times K'$, the distances $d_{D(\rho)}$ converge uniformly to $d_{\mathbb{D}}$ as $\rho \uparrow 1$. We can therefore choose

$$r < \rho_0 < \rho_1 < 1, \qquad 0 < d_1 < \log 3,$$

so that s has no zero on the closed annulus $\rho_0 \le |z| \le \rho_1$ and

$$d_{D(\rho)}(\zeta, \xi) \le d_1 \qquad (\zeta, \xi \in K', \ \rho_0 \le \rho < 1). \tag{5.7}$$

Here we use monotonicity of hyperbolic distance under inclusion of disks. The condition concerning the annulus can be arranged after increasing ρ_0, since the zeros of s are isolated.

Evaluating $a_n = s_n - b_n$ at β_n and $b_n = s_n - a_n$ at α_n gives

$$\|a_n\|_r \ge c_0/2, \qquad \|b_n\|_r \ge c_0/2 \tag{5.8}$$

eventually. Define

$$M_n(t) = \langle \max(0, u_n, v_n) \rangle_t\,, \qquad \epsilon_n = \max_{|z|\le\rho_1} \max\{|a_n/A_n|, |b_n/B_n|\}.$$

On the chosen annulus, for some fixed $c_1 > 0$,

$$c_1 \le |s_n| \le 2\epsilon_n e^{\max(u_n,v_n)}$$

for all large n. Hence $\max(u_n, v_n) \to \infty$ uniformly there, and $M_n(\rho_0) \to \infty$. Lemma 2.4 provides radii

$$\rho_n \in [\rho_0, \rho_1), \qquad M = M_n(\rho_n), \qquad \widehat{\rho}_n = \rho_n + M^{-1} < \rho_1, \qquad M_n(\widehat{\rho}_n) \le 2M.$$

In the rest of the proof M depends on n and tends to infinity. Eventually $|a_n| \le |A_n|$ and $|b_n| \le |B_n|$ on $\overline{D(\rho_1)}$. Lemma 2.3, applied with (5.8), therefore yields

$$m(\rho_n, a_n'/a_n) + m(\rho_n, b_n'/b_n) = O(\log M). \tag{5.9}$$

Let U_n, H_n be the harmonic extensions in $D(\rho_n)$ of the boundary functions $\max(u_n, v_n)$ and $\min(u_n, v_n)$, respectively. Then

$$U_n > 0, \qquad U_n(0) = M, \qquad H_n = u_n + v_n - U_n.$$

Both $P_n = U_n - v_n$ and $Q_n = U_n - u_n$ are nonnegative harmonic functions. Let Γ_n be the hyperbolic geodesic segment in $D(\rho_n)$ joining α_n to β_n. Apply Corollary 5.2 after rescaling the disk, using (5.5) and (5.7). We obtain a fixed $\varepsilon > 0$ such that

$$H_n(z) \le -\varepsilon U_n(z) + 2C_0 \qquad (z \in \Gamma_n). \tag{5.10}$$

A centered hyperbolic ball is geodesically convex, so $\Gamma_n \subset \overline{D(r)}$. Its hyperbolic length is at most d_1, and its Euclidean length is at most $\rho_n d_1/2$. Harnack's inequality and (5.10) give fixed constants $c, C > 0$ such that, for all large n,

$$cM \le U_n \le CM \quad \text{on } \overline{D(r)}, \qquad H_n \le -cM \quad \text{on } \Gamma_n.$$

We extend the last estimate to a neighborhood of fixed width. Set $V_n = U_n - H_n = P_n + Q_n \ge 0$. Uniformly for $w \in \overline{D(r)}$ and $\rho_n \ge \rho_0$, Harnack's inequality gives a factor $q_\delta \downarrow 1$ as $\delta \downarrow 0$ such that

$$U_n(z) \le q_\delta U_n(w), \qquad V_n(z) \ge q_\delta^{-1} V_n(w) \qquad (|z - w| \le \delta).$$

For $w \in \Gamma_n$, it follows that

$$H_n(z) \le (q_\delta - q_\delta^{-1})CM - q_\delta^{-1}cM.$$

Fix $0 < \delta < \rho_0 - r$ sufficiently small. For some $c_2 > 0$,

$$H_n(z) \le -c_2 M \qquad \Big(\operatorname{dist}(z, \Gamma_n) \le \delta\Big) \tag{5.11}$$

for all large n, where dist denotes Euclidean distance.

Consider the holomorphic function

$$W_n = s_n a_n' - s_n' a_n = b_n a_n' - a_n b_n' = -s_n b_n' + s_n' b_n.$$

The functions s_n, s_n' are uniformly bounded on $\overline{D(\rho_1)}$. The first and last expressions for W_n give, almost everywhere on $|z| = \rho_n$,

$$\log|W_n| \le \min(u_n, v_n) + E_n, \qquad E_n = C_1 + \log^+|a_n'/a_n| + \log^+|b_n'/b_n| \ge 0,$$

where C_1 is fixed. The singularities of E_n are integrable, and its mean is $O(\log M)$ by (5.9). Poisson comparison, with kernels uniformly bounded on $\overline{D(r+\delta)}$, gives

$$\log|W_n(z)| \le H_n(z) + O(\log M) \qquad (|z| \le r+\delta).$$

When $W_n \equiv 0$, this inequality is understood in the extended sense. Together with (5.11), it implies

$$\sup_{\operatorname{dist}(z,\Gamma_n)\le\delta} |W_n(z)| \longrightarrow 0.$$

Choose $R_1 \in (r, r+\delta)$ so that s has no zero on $|z| = R_1$. Enclose all its zeros in $D(R_1)$ in pairwise disjoint closed disks contained in $D(R_1)$, disjoint from K', with diameters less than δ and no zeros on their boundaries. Replace the portions of Γ_n inside these disks by boundary arcs. The resulting paths $\widetilde{\Gamma}_n$ have the same endpoints and stay within distance δ of Γ_n. Their lengths are uniformly bounded: the supporting circle or line of the geodesic meets each disk boundary in at most two points, so each replacement adds at most one circumference. All the paths lie in the fixed compact set obtained from $\overline{D(R_1)}$ by removing the interiors of these disks. This set has no zero of s, and therefore $|s_n| \ge c_3 > 0$ there eventually. Integration of $(a_n/s_n)' = W_n/s_n^2$ now gives

$$\left|\frac{a_n(\beta_n)}{s_n(\beta_n)} - \frac{a_n(\alpha_n)}{s_n(\alpha_n)}\right| \le \operatorname{length}(\widetilde{\Gamma}_n) c_3^{-2} \sup_{\widetilde{\Gamma}_n} |W_n| \longrightarrow 0.$$

This contradicts (5.6). □

The proof uses only two failure points in fixed compact subsets of Ω and harmonic comparison in the ambient disk; the joining geodesic need not lie in Ω. In particular, Proposition 5.3 applies on $D(r_*)$ by (5.1). The example below will show that neither this radius nor the diameter $\log 3$ can be increased, even when $s_n = 1$.

The sharp example.

We refine the construction of [Ere96b, Section 4]. Define

$$\Phi(z) = \frac{2z}{\sqrt{1-z^4}}, \qquad L(z) = \frac{1-z^2}{1+z^2} - \frac{1-z}{2(1+z)}, \qquad u = \operatorname{Re} L.$$

The square root is the holomorphic branch equal to one at zero, which exists because $1-z^4$ has no zeros in $\mathbb{D}$. Put $v = -\operatorname{Re}(\Phi^2)$. A calculation gives

$$u - v = \frac{1}{2}\operatorname{Re}\frac{1+z}{1-z} > 0. \tag{5.12}$$

Also $u(z) > 0$ for $\operatorname{Re} z \ge 0$. Indeed, if $z = x+iy$ and $s = |z|^2$, then

$$u(z) = (1-s)\left\{\frac{1+s}{|1+z^2|^2} - \frac{1}{2|1+z|^2}\right\}.$$

Multiplication of the expression inside the braces by $2|1+z^2|^2|1+z|^2$ gives

$$(1+s)^2 + 4x(1+s) + 4y^2 > 0 \qquad (x \ge 0).$$

The real parts of $\Phi(z)$ and z have the same sign. To see this, use

$$\operatorname{Im}\frac{w}{1-w^2} = \frac{(1+|w|^2)\operatorname{Im} w}{|1-w^2|^2}$$

with $w = z^2$. In the right half-disk, a zero of $\operatorname{Re}\Phi$ would force Φ^2 to be real and nonpositive. The displayed identity rules this out off the real axis, and Φ is positive on the positive real axis. Connectedness gives the sign throughout the right half-disk. Oddness treats the left half-disk, and Φ is purely imaginary on the imaginary axis.

For positive integers n, let

$$G_n(w) = \frac{1}{2} + \sqrt{\frac{n}{\pi}} \int_0^w e^{-n\zeta^2}\, d\zeta.$$

The integral defines an entire function, and $G_n(w) + G_n(-w) = 1$. For $w = x + iy$ with $x \le 0$, shifting the integration contour to a horizontal line gives

$$|G_n(w)| \le \sqrt{\frac{n}{\pi}} e^{ny^2} \int_{-\infty}^{x} e^{-nt^2}\, dt \le \frac{1}{2} e^{-n\operatorname{Re}(w^2)}.$$

The vertical segment at real part $-T$ contributes zero as $T \to \infty$. For the last inequality, write $t = x - \tau$, $\tau \ge 0$, and use $(x-\tau)^2 \ge x^2 + \tau^2$. Symmetry gives, for $x \ge 0$,

$$|G_n(w)| \le 1 + \frac{1}{2} e^{-n\operatorname{Re}(w^2)}.$$

Set

$$a_n = G_n \circ \Phi, \qquad b_n = 1 - a_n, \qquad A_n = 2ne^{nL}, \qquad B_n(z) = A_n(-z).$$

Since Φ is odd, $b_n(z) = a_n(-z)$. Its sign property, the positivity of u in the right half-disk, and (5.12) imply

$$|a_n(z)| \le \frac{3}{2} e^{nu(z)}, \qquad |b_n(z)| \le \frac{3}{2} e^{nu(-z)}.$$

Consequently,

$$\sup_{\mathbb{D}} |a_n/A_n| \le \frac{3}{4n}, \qquad \sup_{\mathbb{D}} |b_n/B_n| \le \frac{3}{4n}. \tag{5.13}$$

Thus the five functions

$$A_n, \qquad B_n, \qquad a_n - A_n, \qquad b_n - B_n, \qquad -1 \tag{5.14}$$

are units with zero sum for every $n \ge 1$.

At zero, $A_n(0) = B_n(0) = 2ne^{n/2} \to \infty$. On the negative real axis,

$$L(-t) = \frac{(1+t)(1-4t+t^2)}{2(1-t)(1+t^2)} \qquad (0 < t < 1).$$

This expression is negative precisely for $t > r_*$. Hence, for every $r_* < R \le 1$, choosing $r_* < t < R$ gives

$$A_n(-t) \longrightarrow 0, \qquad B_n(t) \longrightarrow 0. \tag{5.15}$$

We now prove Theorem 1.3 and the optimality assertions following it.

Proof of Theorem 1.3. Apply Cartan's theorem to the given sequence of five units. After extraction, either all indices form a C-class, or there are two disjoint C-classes. In the latter case, if these classes do not cover all five indices, both have size two. Relabel and divide all functions by $-f_{5,n}$ so that $f_{5,n} = -1$ and the two classes are $\{1,3\}$ and $\{2,4\}$, with dominant indices 1 and 2, respectively. This reduction is the one used in [Ere96b].

Put

$$A_n = f_{1,n}, \qquad B_n = f_{2,n}, \qquad a_n = f_{1,n} + f_{3,n}, \qquad b_n = f_{2,n} + f_{4,n}.$$

The definition of C-classes gives $a_n/A_n \to 0$ and $b_n/B_n \to 0$, while $a_n + b_n = 1$. Proposition 5.3 applies on Ω. Whichever reciprocal tends to zero allows the constant function $f_{5,n}$

to be added to its corresponding pair: its quotient tends to zero, and so does the normalized sum of the enlarged class. The two resulting classes form the required partition.

To prove optimality, fix $r_* < R \leq 1$ and consider (5.14). The first four functions tend in modulus to infinity at zero. By (5.13) and (5.15), each tends to zero at one of the points $-t, t$ in $D(R)$. A class containing the constant -1 cannot have one of these four indices as a dominant index, since the quotient of the constant would be unbounded at that point. Nor can the index of the constant be a dominant index of a class containing any of the other functions, since their quotients are unbounded at zero. The singleton consisting of the constant is not a C-class. These obstructions persist under every extraction, so no subsequence has a C-class partition on $D(R)$.

By (5.1), the positive assertion on $D(r_*)$ and these examples give $R_5 = 2 - \sqrt{3}$. For any proposed universal diameter larger than $\log 3$, choose $R > r_*$ sufficiently close to r_* that $4 \operatorname{arctanh} R$ is smaller than that diameter. The same example on $\Omega = D(R)$ proves that $\log 3$ is optimal. $\square$

6. Zero directions of the Kobayashi–Royden pseudometric

We apply Theorem 1.2 to holomorphic functions on the unit disk obtained by pulling back the monomials of a Laurent polynomial. The point is that a map from $\mathbb{D}$ to $(\mathbb{C}^*)^N$ has zero-free coordinate functions, so both positive and negative powers are holomorphic on $\mathbb{D}$.

Proof of Theorem 1.4. Let $Y \subset (\mathbb{C}^*)^N$ be as in Theorem 1.4. If $\gamma_{x,\boldsymbol{v}}(\mathbb{C}) \subset Y$, the discs $z \mapsto \gamma_{x,\boldsymbol{v}}(nz)$ show that $\kappa_Y(x; \boldsymbol{v}) = 0$. Conversely, suppose that $\kappa_Y(x; \boldsymbol{v}) = 0$. There is nothing to prove when $\boldsymbol{v} = 0$. Otherwise choose $\boldsymbol{F}_n \in \mathcal{O}(\mathbb{D}, Y)$ such that

$$\boldsymbol{F}_n(0) = x, \qquad \boldsymbol{F}_n'(0) = t_n \boldsymbol{v}, \qquad t_n \to \infty.$$

Write $\boldsymbol{F}_n = (F_{1,n}, \dots, F_{N,n})$.

Choose finitely many nonzero Laurent polynomials defining Y in the torus; for $Y = (\mathbb{C}^*)^N$ there are no equations to check. Consider one of them, written with distinct exponent vectors as

$$P(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha} \in E} c_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}, \qquad c_{\boldsymbol{\alpha}} \neq 0, \qquad E \subset \mathbb{Z}^N,$$

where

$$\boldsymbol{z}^{\boldsymbol{\alpha}} = \prod_{j=1}^{N} z_j^{\alpha_j}, \qquad \boldsymbol{F}_n^{\boldsymbol{\alpha}} := \prod_{j=1}^{N} F_{j,n}^{\alpha_j}.$$

Every $F_{j,n}$ is a unit on $\mathbb{D}$, so $\boldsymbol{F}_n^{\boldsymbol{\alpha}}$ is a unit even when some α_j are negative. Since $P(\boldsymbol{F}_n(z)) = 0$ for every $z \in \mathbb{D}$, the functions

$$f_{\boldsymbol{\alpha},n} := c_{\boldsymbol{\alpha}} \boldsymbol{F}_n^{\boldsymbol{\alpha}}, \qquad \boldsymbol{\alpha} \in E,$$

are units on $\mathbb{D}$ and their sum is identically zero. If $|E| = 1$, this is impossible because a single nonzero monomial is a unit. If $|E| = 2$, the quotient of the two terms is identically -1, so the two indices already form a C-class. Thus only the case $|E| \geq 3$ requires Theorem 1.2.

Let $I \subset E$ be one of the resulting C-classes, and let $\boldsymbol{\beta} \in I$ be a dominant index. Evaluating its normalized sum at 0 and using $\boldsymbol{F}_n(0) = x$ gives

$$\sum_{\boldsymbol{\alpha} \in I} c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} = 0.$$

The quotients $f_{\boldsymbol{\alpha},n}/f_{\boldsymbol{\beta},n}$ are locally bounded on a fixed disk about zero. Cauchy's estimate applied to these quotients at 0 gives

$$t_n \frac{c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}}{c_{\boldsymbol{\beta}} x^{\boldsymbol{\beta}}} (\boldsymbol{\alpha} - \boldsymbol{\beta}) \cdot \boldsymbol{\lambda} = O(1), \qquad \boldsymbol{\alpha} \in I,$$

where $\boldsymbol{\alpha} \cdot \boldsymbol{\lambda} = \sum_{j=1}^N \alpha_j \lambda_j$ and $\lambda_j = v_j/x_j$. Since $t_n \to \infty$, it follows that $(\boldsymbol{\alpha} - \boldsymbol{\beta}) \cdot \boldsymbol{\lambda} = 0$ for every $\boldsymbol{\alpha} \in I$. Hence $\boldsymbol{\alpha} \cdot \boldsymbol{\lambda}$ has a common value ν_I on I, and

$$P(\gamma_{x,\boldsymbol{v}}(\zeta)) = \sum_I e^{\nu_I \zeta} \sum_{\boldsymbol{\alpha} \in I} c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} = 0.$$

Repeating this argument for the finitely many defining polynomials proves that $\gamma_{x,\boldsymbol{v}}$ takes its values in Y. □

The same argument also controls sequences of tangent vectors with varying base points. Write

$$\mathcal{N}_Y = \{(x, \boldsymbol{v}) \in TY : \kappa_Y(x; \boldsymbol{v}) = 0\}.$$

Proposition 6.1. *For every smooth closed algebraic subvariety $Y \subset (\mathbb{C}^*)^N$, the set $\mathcal{N}_Y$ is a closed algebraic subset of TY. For every compact set $K \subset TY \setminus \mathcal{N}_Y$,*

$$\inf_{(x,\boldsymbol{v}) \in K} \kappa_Y(x; \boldsymbol{v}) > 0.$$

Proof. Consider a defining Laurent polynomial $P(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha} \in E} c_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$, and put $q = |E|$. For $\lambda_j = v_j/x_j$, the identity $P \circ \gamma_{x,\boldsymbol{v}} = 0$ is equivalent to the finite set of equations

$$\sum_{\boldsymbol{\alpha} \in E} c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} (\boldsymbol{\alpha} \cdot \boldsymbol{\lambda})^k = 0, \qquad 0 \le k < q.$$

Indeed, group the terms by the distinct values of $\boldsymbol{\alpha} \cdot \boldsymbol{\lambda}$. If there are d such values, the first d equations form a Vandermonde system for the grouped coefficients. They force every coefficient to vanish, which proves the identity; the reverse implication follows by differentiation at zero. These are regular algebraic equations in $(x, \boldsymbol{v})$ on the tangent bundle of the torus. Apply them to each of the finitely many defining polynomials and use Theorem 1.4. This proves the assertion about $\mathcal{N}_Y$.

Suppose, contrary to the second assertion of the proposition, that for some compact set $K \subset TY \setminus \mathcal{N}_Y$ one has $\inf_K \kappa_Y = 0$. Then there are $(x_n, \boldsymbol{v}_n) \in K$ converging, after passage to a subsequence, to $(x, \boldsymbol{v}) \in K$, and discs $\boldsymbol{F}_n \in \mathcal{O}(\mathbb{D}, Y)$ with

$$\boldsymbol{F}_n(0) = x_n, \qquad \boldsymbol{F}_n'(0) = t_n \boldsymbol{v}_n, \qquad t_n \to \infty.$$

For each defining polynomial apply Theorem 1.2 to $c_{\boldsymbol{\alpha}} \boldsymbol{F}_n^{\boldsymbol{\alpha}}$, with the cases of one or two terms treated as above, and extract successively for the finitely many polynomials. In each C-class I, choose a dominant index $\boldsymbol{\beta}$. Then evaluation of the normalized sum gives

$$\sum_{\boldsymbol{\alpha} \in I} c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}} = 0.$$

Put $\lambda_{j,n} = v_{j,n}/x_{j,n}$, so that $\boldsymbol{\lambda}_n \to \boldsymbol{\lambda} = (v_j/x_j)_{j=1}^N$. Bounded quotient derivatives give

$$t_n \frac{c_{\boldsymbol{\alpha}} x_n^{\boldsymbol{\alpha}}}{c_{\boldsymbol{\beta}} x_n^{\boldsymbol{\beta}}} (\boldsymbol{\alpha} - \boldsymbol{\beta}) \cdot \boldsymbol{\lambda}_n = O(1), \qquad \boldsymbol{\alpha} \in I.$$

The fraction converges to a nonzero number. Dividing by t_n and passing to the limit therefore yields $(\boldsymbol{\alpha} - \boldsymbol{\beta}) \cdot \boldsymbol{\lambda} = 0$. The proof of Theorem 1.4 now gives $\gamma_{x,\boldsymbol{v}}(\mathbb{C}) \subset Y$, contrary to $(x, \boldsymbol{v}) \notin \mathcal{N}_Y$. □

We next specialize to projective space with hyperplanes omitted. Following the symmetric coordinates of Kiernan and Kobayashi [KK73, pp. 208–209], write

$$H_p = \left\{ [x_1 : \cdots : x_p] \in \mathbb{P}^{p-1} : \sum_{j=1}^{p} x_j = 0 \right\}, \qquad X_p = H_p \setminus \bigcup_{j=1}^{p} \{x_j = 0\}, \qquad p \geq 3.$$

Thus $H_p \simeq \mathbb{P}^{p-2}$, and the omitted hyperplanes are in general position. A tuple of units with zero sum defines a holomorphic map to X_p. Conversely, the global affine chart $x_p = 1$ represents every holomorphic map to X_p by such a tuple.

For a nonempty subset $I \subset \{1, \ldots, p\}$, define

$$\pi_I : X_p \longrightarrow \mathbb{P}^{|I|-1}, \qquad \pi_I([x]) = [x_j]_{j \in I}, \qquad H_I = \left\{ [w_j]_{j \in I} : \sum_{j \in I} w_j = 0 \right\}.$$

When $|I| = 1$, the set H_I is empty. For a partition $\mathcal{P} = \{I_1, \ldots, I_q\}$ of $\{1, \ldots, p\}$, put

$$\Pi_{\mathcal{P}} = (\pi_{I_1}, \ldots, \pi_{I_q}), \qquad Y_{\mathcal{P}} = \prod_{\nu=1}^{q} H_{I_\nu}.$$

Proposition 6.2. *Fix $p \geq 3$ and $0 < R \leq 1$. The C-class partition property on $D(R)$ is equivalent to the following assertion: every sequence of holomorphic maps $F_n : \mathbb{D} \to X_p$ has a subsequence and a partition $\mathcal{P}$ such that $\Pi_{\mathcal{P}} \circ F_n$ converges locally uniformly on $D(R)$ to a holomorphic map into $Y_{\mathcal{P}}$.*

Proof. Represent F_n by a tuple $(f_{1,n}, \ldots, f_{p,n})$ of units with zero sum. In a C-class I with dominant index k, the boundedness on compact subsets implies, after extraction of a subsequence,

$$f_{j,n}/f_{k,n} \longrightarrow g_j \quad (j \in I), \qquad g_k = 1, \qquad \sum_{j \in I} g_j = 0.$$

Hence $\pi_I \circ F_n \to [g_j]_{j \in I} \in H_I$. A common subsequence works for the finitely many C-classes.

Conversely, suppose that $\pi_I \circ F_n$ converges to a holomorphic map $G_I : D(R) \to H_I$. Each approximating map avoids the coordinate hyperplanes. Hurwitz's theorem implies that G_I either avoids each such hyperplane or is contained in it [KK73, Proposition 2, pp. 204–205]. Choose $k \in I$ for which the image of G_I is not contained in $\{w_k = 0\}$. It therefore avoids that hyperplane. In this affine chart the quotients $f_{j,n}/f_{k,n}$ converge locally uniformly, and their sum tends to zero. Thus I is a C-class. $\square$

At a point $x = [x_1 : \cdots : x_p] \in X_p$, call a partition $\mathcal{P} = \{I_1, \ldots, I_q\}$ admissible if $\sum_{j \in I_\nu} x_j = 0$ for every ν. This condition is independent of the choice of homogeneous coordinates. Denote the finite set of admissible partitions by $\mathfrak{P}(x)$. Each such partition defines a torus

$$\mathcal{T}_{\mathcal{P},x} = \left\{ [t_{\nu(j)} x_j]_{j=1}^{p} : t_1, \ldots, t_q \in \mathbb{C}^* \right\} \subset X_p,$$

where $\nu(j)$ is the unique index such that $j \in I_{\nu(j)}$. Simultaneous scaling of all t_ν does not change the projective point, so this torus is isomorphic to $(\mathbb{C}^*)^{q-1}$. The one-part partition gives the point x.

Corollary 6.3. *For every $x \in X_p$,*

$$\{ \boldsymbol{v} \in T_x X_p : \kappa_{X_p}(x; \boldsymbol{v}) = 0 \} = \bigcup_{\mathcal{P} \in \mathfrak{P}(x)} T_x \mathcal{T}_{\mathcal{P},x}.$$

Proof. Use the chart $x_p = 1$, in which X_p is the smooth closed subvariety of $(\mathbb{C}^*)^{p-1}$ defined by $x_1 + \cdots + x_{p-1} + 1 = 0$. Represent $\boldsymbol{v}$ by $(v_1, \ldots, v_{p-1}, 0)$ and put $\lambda_j = v_j/x_j$ for $j < p$ and $\lambda_p = 0$. By Theorem 1.4, the equality $\kappa_{X_p}(x; \boldsymbol{v}) = 0$ holds precisely when

$$\sum_{j=1}^{p} x_j e^{\lambda_j \zeta} = 0 \qquad (\zeta \in \mathbb{C}).$$

Group the indices according to the distinct values of λ_j. The linear independence of exponentials with distinct exponents shows that the coordinate sum over each group is zero. These groups form an admissible partition, and the rates λ_j are constant on each part. This is exactly the condition that $\boldsymbol{v}$ belong to the tangent space of the corresponding torus. Conversely, a tangent vector to an admissible torus has rates constant on each part of the corresponding partition, and the same displayed identity follows from their coordinate sums. Theorem 1.4 completes the proof. $\square$

Every part of an admissible partition contains at least two indices. Thus the zero directions at x form a finite union of complex linear subspaces, each of dimension at most $\lfloor p/2 \rfloor - 1$. Corollary 6.3 describes the zero directions, including those over the diagonal hyperplanes $\sum_{j \in I} x_j = 0$, $2 \leq |I| \leq p - 2$. This description supplements the positivity of the Kobayashi–Royden pseudometric away from their union established in [KK73, Theorem 3 and Theorem 6].

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Mathematics Department, Purdue University, West Lafayette, IN 47907, USA
Email address: `eremenko@purdue.edu`

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China
Email address: `zbxu@mail.xjtu.edu.cn`

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China
Email address: `teng.zhang@stu.xjtu.edu.cn`