Solution of 7.9.1 with the method of undetermined coefficients

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f},$$

where

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} e^t \\ t \end{pmatrix} = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The characteristic equation is

$$\lambda^2 - 1 = 0,$$

and we found eigenvalues and eigenvectors which give the solution of the homogeneous system $\mathbf{x}' = Ax$,

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

According to the method of undetermined coefficients we should look now for a partial solution in the form

$$\mathbf{x}(t) = \mathbf{a}te^t + \mathbf{b}e^t + \mathbf{c}t + \mathbf{d}.$$

Substituting this we obtain

$$\mathbf{a}te^t + \mathbf{a}e^t + \mathbf{b}e^t + \mathbf{c} = A(\mathbf{a}te^t + \mathbf{b}e^t + \mathbf{c}t + \mathbf{d}) + \mathbf{f}$$

This gives a system

$$A\mathbf{a} = \mathbf{a}, \tag{1}$$

$$A\mathbf{b} = \mathbf{b} + \mathbf{a} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2}$$

$$A\mathbf{c} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3}$$

$$A\mathbf{d} = \mathbf{c}, \tag{4}$$

by writing separate equalities for terms with te^t, e^t, t and the constant term.

The system decouples: from the first two equations $\mathbf{a}$ and $\mathbf{b}$ can be found; and from (3) and (4) one can find $\mathbf{c}$ and $\mathbf{d}$. We found in class that

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Now we treat (1) and (2).

Equation (1) says that $\mathbf{a}$ is an eigenvector of A with eigenvalue 1. In the beginning we found one eigenvector $(1,1)^T$. *We also found that with THIS eigenvector, equation (2) for* $\mathbf{b}$ *has no solutions. But we did not have to take this eigenvector. General solution of (1) has the form* $\mathbf{a} = (s,s)^T$*, where* s *is arbitrary. Using it, (2) becomes*:

$$\begin{aligned} b_1 - b_2 &= s - 1, \\ 3b_1 - 3b_2 &= s. \end{aligned}$$

This has solution if and only if $3(s-1) = s$, that is $s = 3/2$. And one can take $b_1 = 3/2,\ b_2 = 0$. Therefore, the complete answer is

$$\mathbf{x}(t) = \frac{3}{2}te^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{2}e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} + c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$