Let

$$I = \int_0^\infty \frac{dx}{x^3+1}.$$

Following the hint, we integrate $f(z) = 1/(z^3+1)$ over the boundary of the sector

$$S_\rho = \{re^{i\theta} : 0 < \theta < 2\pi/3,\ 0 < r < \rho\}.$$

The integral over the circular arc tends to 0, so when $\rho \to +\infty$

$$\int_{\partial S_\rho} f(z)dz \to I(1 - e^{2\pi i/3}).$$

On the other hand, S_ρ contains one simple pole of our function f, at the point $a := e^{\pi i/3}$. So

$$\int_{\partial S_\rho} f(z)dz = 2\pi i \operatorname{res}_a f(z) = \frac{1}{3e^{2\pi i/3}}.$$

So

$$I = \frac{2\pi i e^{-2\pi i/3}}{3(1 - e^{2\pi i/3})} = \frac{\pi}{3\sin(\pi/3)} = \frac{2\pi}{3\sqrt{3}}.$$