Problem 6 from the practice exam

Use the formula from HW Problem 18 on p. 328. It involves a sum from $-\infty$ to ∞, so we write:

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{1}{2}\left(\sum_{-\infty}^{\infty} \frac{(-1)^n}{1+n^2} - 1\right).$$

And using this formula,

$$\sum_{-\infty}^{\infty} \frac{(-1)^n}{1+n^2}$$

equals to the sum of the residues at the two poles of $f(z) = 1/(1+z^2)$. f has two simple poles, at i and $-i$. Computing the residues:

$$\mathrm{res}_i \frac{\pi}{(1+z^2)\sin \pi z} = \mathrm{res}_i \frac{\pi}{(z+i)(z-i)\sin \pi z} = \frac{\pi}{2i \sin \pi i}.$$

Similarly,

$$\mathrm{res}_{-i} = \frac{\pi}{(-2i)\sin(-\pi i)} = \frac{\pi}{2i \sin \pi i}.$$

So

$$S = \frac{1}{2}\left(\frac{2\pi}{2i \sin \pi i} - 1\right) = (\pi/\sinh \pi - 1)/2.$$