Complex numbers and trigonometry

September 23, 2019

1. Compute
$$\left(\frac{1-i}{1+i}\right)^{10}.$$
(To compute means to express as $a+bi$, where a and b are real numbers, expressed as simple as possible.)

2. Find all four solutions of $z^4 = -1$.

3. Compute the first five significant digits of $\sin 1^\circ$, using only paper and pencil (no calculators, no computers, no books, no Internet). You may use $\pi \approx 3.1415927$. (This computation was performed for the first time by Ptolemy in 2-nd century AD. But he did not know everything that you learned in this class.)

4. A wire is stretched tight between two points in the plane at a distance 1000 feet. Then 1 inch is added to the wire, so that it becomes a bit loose, and the middle is lifted as much as possible over the plane. Can a cat pass through the gap between the plane and the wire? Can a car pass under the wire?

5. Solve the equation $z^3 = 1$ in radicals, that is without using trigonometric or exponential functions, only roots of rational numbers.

6. Solve the equation $z^5 = 1$ in radicals.

7. The map $z \mapsto \overline{z}$ is a reflection in the real axis. Find a formulas for the reflection in the imaginary axis, and in the line $\Re z = \Im z$. Your formulas must be of the form $z \mapsto c\overline{z}$ with an appropriate c.

8. Express $\cos(5t)$ in terms of $\cos t$.

9. Express $\sin(4t)$ in terms of $\cos t$ and $\sin t$.

10. Use Euler's formula to find a closed form expression for the sum

$$\sin t + \sin(2t) + \ldots + \sin(nt).$$

Your answer should contain only real numbers when t is real.

11. Use Euler's formula to find a closed form expression for the sum

$$1 + \cos t + \cos(2t) + \ldots + \cos(nt).$$

12. Find all complex solutions of the equation

$$\cos z = 2,$$

and make a picture of them.

13. If complex numbers are interpreted as vectors in the plane, what is the geometric meaning of the quantity $\Re(z_1\overline{z_2})$?

14. Solutions of a cubic equation

$$z^3 = az + b = 0.$$

are given by Cardano's formula

$$z = \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} + \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}}.$$

Does this formula give a correct answer for $a = 1$ and $b = 0$?