Practice problems set 1

March 1, 2021

1. Inspect table 1 on p. 26–28 and tell which of the Fourier series 1–20 converge *uniformly* on the interval where the function in the left column is defined.

Solution. Theorem 2.5 on p. 41 says that for uniform convergence it is sufficient that the periodic function be continuous and piecewise smooth. On the other hand, if a function is discontinuous then convergence cannot be uniform. Inspection of the graphs on the right hand side shows that uniformly convergent examples are 2, 5, 8, 9, 10, 11, 15, 16, 17.

2. Let $f(x) = e^x$, $0 \leq x \leq \pi$, and let $F(x)$ be the sum of its sine Fourier series. Find $F(0)$.

Solution. $F(x)$ is the odd periodic extension of f, therefore $F(0) = (F(0+) + F(0-))/2 = 0$, by Theorem 2.1, p. 35.

3. Which of these functions belongs to $L^2(-\infty, +\infty)$:
a) $(\sin x)/x$,
b) $(\sin\sqrt{|x|})/\sqrt{|x|}$,
c) $x^2 e^{-|x|}$,
d) $x/(1+x^2)$.

Answer. a), c), d).

4. Consider the following sequences of functions $f_n(x) = x^n$ and $g_n(x) = \sqrt{n}x^n$, defined on the interval $(0,1)$. Which of these two sequences converge to 0

a) at every point of the interval $(0, 1)$
b) uniformly on the interval $(0, 1)$.
c) in the sense of $L^2(0, 1)$.

Answer. a) both, b) none, c) only f_n.

5. Consider the sequence of functions $f_n(x) = e^{-|x-n|}$ on the whole real line. As $n \to +\infty$, does this sequence converge:
a) at every real point?
b) uniformly on the whole real line?
b) in $L^2(-\infty, +\infty)$?

Answer. a) yes, b) no, c) no.

6. Which of the following sequences are sine Fourier coefficients of some function $f \in L^2(0, 1)$?
a) $b_n = 1/n$
b) $b_n = 1/\sqrt{n}$
c) $b_n = 1/(\sqrt{n}\log(n + 1))$.

Ans. According to the Riesz-Fisher theorem, the necessary and sufficient condition for a sequence to be Fourier coefficients of an L^2 function is

$$\sum_{n=1}^{\infty} |b_n|^2 < \infty.$$

Therefore sequences a) and c) are Fourier coefficients of some functions in $L^2(0, 1)$ while sequence b) is not since the series $\sum 1/n$ diverges. For c), compare the sum of $|b_n|^2$ with the integral

$$\int^{\infty} \frac{dx}{x \log^2 x} = \int^{\infty} \frac{dy}{y^2} < \infty.$$

7. Using Table 1 on p. 26-28, find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2}.$$

Solution. There are many ways to do this. The simplest one is probably to use entry 6 of the table. Write it in the form

$$1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}, \quad 0 < x < \pi.$$

Since the functions in the right hand side are orthogonal, Pythagoreans tell us that

$$\|1\|_2^2 = \frac{16}{\pi^2} \sum_{n=0}^{\infty} \frac{\|\sin(2n+1)x\|_2^2}{(2n+1)^2}.$$

Since for all n

$$\|\sin(nx)\|_2^2 = \int_0^{\pi} \sin^2(nx)dx = \pi/2,$$

and

$$\|1\|^2 = \pi,$$

we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

To check the result, compare it with Exercise 2 (4) on p. 37. It says

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

So the sum of even terms of this series is $\pi^2/24$, and sum of odd terms must be $\pi^2(1/6 - 1/24) = \pi^2/8$.