Projectors

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1. An operator $P : V \to V$ is called *idempotent* if

$$P^2 = P. \tag{1}$$

Let N_P be the null-space and R_P the range (column space). Then

$$N_P \cap R_P = \{0\}. \tag{2}$$

Indeed, suppose that $x \in N_P \cap R_P$. This means that $Px = 0$ and $x = Py$ for some $y \in V$. Then

$$x = Py = PPy = Px = 0.$$

Then

Every vector $x \in X$ can be represented as a sum $x = u + v$, where $u \in R_P$ and $v \in N_p$.

Indeed, we can take $u = Px$ and $v = x - Px$. Then $u \in R_p$ and $v \in N_P$ because $Pu = Px - PPx = 0$. Uniqueness of representation follows from (2).

Whenever we have k subspaces U_j with the property that every vector in V can be uniquely represented as a sum $u_1 + \ldots + u_k$ where $u_j \in U_j$, we say that V is a *direct sum* of the U_j, and write

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_k. \tag{3}$$

So for every idempotent P we have

$$V = N_P \oplus R_P.$$

It follows that all idempotents are diagonalizable, with eigenvalues 0 and 1. Indeed, N_P is the eigenspace corresponding to eigenvalue 0, and R_P is the eigenspace corresponding to eigenvalue 1. Choose a basis of the space V consisting of a basis in R_P and a basis in N_P. It consists of eigenvectors.

If P is idempotent then $I - P$ is also idempotent.

Indeed,

$$(I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P.$$

2. An idempotent which is Hermitian is called a *projector*. So in addition to (1) projectors satisfy

$$P^* = P, \tag{4}$$

where the star denoted Hermitian conjugation (transposition and complex conjugation). An idempotent is a projector if and only if $N_P \perp R_P$. Indeed, if $x \in N_P$ and $y \in R_P$, that is $Px = 0$ and $y = Pw$ then

$$(x, y) = (x, Pw) = (Px, w) = (0, w) = 0.$$

Here we only used that P is Hermitian, so for every Hermitian operator A, the null space and the column space are orthogonal.

There is a one-to-one correspondence between projectors and subspaces $U \subset V$. To each projector we put into correspondence the subspace $R_P = P(V)$.

And conversely, for each subspace U we have the direct decomposition

$$V = U \oplus U^\perp,$$

so each $x \in V$ can be written as $x = u + v$, where $u \in U,\ v \in U^\perp$ and we define $Px = u$. Then $R_P = U$.

Let us check that our new definition is equivalent to the old definition of the projector. According to the old definition, a projector onto the space U is defined by the property that

$$(x - Px) \perp w \quad \text{for every} \quad w \in U, \tag{5}$$

and $P = A(A^*A)^{-1}A^*$, for some matrix A. Now it is easy to check that P so defined satisfy (1) and (4). Conversely, if P satisfies (1) and (4) then we have a direct decomposition $V = R_P \oplus N_P$, where $R_P \perp N_P$. So if we write $x = u + v,\ u \in R_P, v \in N_P$ and define $Px = u$, then $x - Px = v$ is indeed perpendicular to every $w \in R_P$, and we obtain (5).

3. *Let P_1, P_2 be two projectors, $U_1 = P_1(V),\ U_2 = P_2(V)$. Then $U_1 \perp U_2$ if and only if $P_1P_2 = P_2P_1 = 0$.*

Proof. Let $x_i = P_iy_i,\ i \in \{1,2\}$ be arbitrary elements of U_i. Then

$$(x_1, x_2) = (P_1y_1, P_2y_2) = (P_2P_1y_1, y_2) = (y_1, P_1P_2y_2).$$

Then $U_1 \perp U_2$ if and only if this is zero for each y_1, y_2. This happens if and only if $P_1P_2 = P_2P_1 = 0$.

Again, we did not use that P_1 and P_2 are projectors: for any two Hermitian operators A and B the column spaces are orthogonal if and only if $AB = BA = 0$.

Corollary. For Hermitian operators $AB = 0$ implies $BA = 0$.

Exercise. Show by example that this is not true for arbitrary square matrices.

4. Partition of unity. Suppose that V is represented as a direct sum (3) of k orthogonal subspaces U_j, and let P_j be projectors onto U_j. Then we have

$$I = P_1 + \dots P_k, \quad P_iP_j = 0. \tag{6}$$

And vice versa: if (6) holds and $U_j = R_{P_j}$ then we have (3) and $U_i \perp U_j$. Such a family of projectors is called a *partition of unity.*

For example, let A be a Hermitian (or more generally, normal) operator. Let λ_j be all its eigenvalues, and U_j the corresponding eigenspaces.Then U_j are orthogonal to each other and (3) holds. So

To every normal operator A a partition of unity corresponds such that

$$A = \lambda_1P_1 + \dots + \lambda_kP_k.$$

5. Permutable projectors.

The product of two projectors is a projector if and only if they commute.

Proof. First of all a product of Hermitian operators is Hermitian if and only if they commute. Indeed, let A and B be Hermitian, then

$$(AB)^* = B^*A^* = BA.$$

This is equal to AB iff A and B commute.

Now suppose that P_1 and P_2 are projectors, $P_1P_2 = P_2P_1$, and $P = P_1P_2$. Then

$$P^2 = P_1P_2P_1P_2 = L_1^2P_2^2 = P_1P_2 = P,$$

so we have (1), and

$$P^* = (P_1P_2)^* = P_2^*P_1^* = P_2P_1 = P_1P_2 = P,$$

so we have (4).

Exercise. Suppose that we have two commuting projectors, $P_1P_2 = P_2P_1$. *Let* $U_i = P_i(V)$.

Then P_1P_2 *is the projector onto* $U_1 \cap U_2$*, and*

$$I - (I - P_1)(I - P_2) = P_1 + P_2 - P_1P_2$$

projects onto the span $U_1 + U_2 := \{u_1 + u_2 : u_1 \in U_1,\ u_2 \in U_2\}$.

6. *Orthoprojectors decrease length.*

First we prove a useful formula

$$\|Px\|^2 = (Px, Px) = (P^2x, x) = (Px, x). \tag{7}$$

Second, $(x - Px) \perp Px$ so

$$\|x\|^2 = \|Px\|^2 + \|x - Px\|^2 \geq \|Px\|^2.$$

This property actually characterizes projectors among all idempotents:

Exercise. Let P be idempotent, $P^2 = P$. Then $\|Pz\| \leq \|x\|$ for all x if and only if P is Hermitian.

Proof. Suppose that some vector $u \in R_P$ is not orthogonal to N_P, and v is the projection of u onto N_P. Then $\|u - v\| < \|u\|$ and $u = P(u - v)$, so $\|u - v\| < \|P(u - v)\|$, contradiction.

7. *A sum*

$$P = P_1 + \ldots, +P_m$$

of projectors is a projector if and only if the summands are orthogonal to each other, that is $P_iP_j = 0,\ i \neq j$.

Proof. It is clear that the sum is Hermitian. Now, if $P_iP_j = 0$ for $i \neq j$, then

$$P^2 = (P_1 + \ldots + P_m)^2 = P_1^2 + \ldots + P_m^2 = P_1 + \ldots + P_m = P.$$

Now suppose that P is idempotent. Fix a $k \in \{1, \ldots, m\}$ and consider $x \in R_{P_k}$ that is $x = P_kx$. Then, using (7),

$$\|x\|^2 = \|P_kx\|^2 \leq \sum_{j=1}^{m} \|P_jx\|^2 = \sum_{j=1}^{n} (P_jx, x) = (Px, x) = \|Px\|^2 = \|x\|^2.$$

this implies that $P_jx = 0$ for all $j \neq k$. So $P_jP_k = 0$ for all $j \neq k$.

8. *A difference* $P = P_1 - P_2$ *of projectors is a projector if and only if it is idempotent.*

Indeed, a sum or a difference of Hermitian operators is always Hermitian.

Corollary. If P is a projector then $I - P$ is a projector. If P projects onto U then $I - P$ projects onto $U^{\perp}$.

9. Let $U_j,\ 1 \leq j \leq m$ be an arbitrary finite collection of subspaces, let P_j be projectors onto the U_j, and let P be the projector on the intersection $\cap_j U_j$. In general, P is not the product of P_j (see section 4). However we have the following formula

$$P = \lim_{k \to \infty} \left(\prod_{j=1}^{m} P^j \right)^k,$$

in the pointwise sense.

10. An operator A is called an *involution* if

$$A^2 = I. \tag{8}$$

Exercise. Prove that A is an involution if and only if

$$A = I - 2P, \tag{9}$$

where P is an idempotent.

For any involution operator A we define the *fixed subspace*

$$F_A = \{x : Rx = x\}.$$

If an involution is Hermitian, and F_A is non-trivial, then it is called a reflection. This happens if and only if P in (9) is a projector onto a non-trivial subspace.

As all Hermitian operators, reflections are diagonalizable, their eigenvalues can be only ± 1 in view of (9) and the theorem of mapping of spectra. So it is also unitary.

Exercise. Prove that all operators which are Hermitian and unitary are reflections.

The subspace F_A is the eigenspace with eigenvalue 1, and the orthogonal complement F_A^T is the eigenspace corresponding to -1.

A reflection with respect to a hyperplane perpendicular to a unit vector c is

$$Rx = x - 2(c, x).$$