An estimate for spherically p-valent functions

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Let us call a meromorphic function in a region D spherically p-valent if the spherical area of the image of this function, counting multiplicity, is at most πp. In other words,

$$\int_D (f^\#)^2(z) dm_z \leq \pi p, \quad \text{where} \quad f^\# = \frac{|f'|}{(1+|f|^2)}.$$

This class is related to mean p-valent holomorphic functions as defined in [1].

Theorem 1. *The family of all spherically p-valent functions in a region D is quasinormal of order p in D.*

Proof. We may assume without loss of generality that D is the unit disc. Let f_n be a sequence of spherically p-valent functions. Write $f_n = g_n/h_n$ so that g_n and h_n have no common zeros. Let

$$u_n = \log\sqrt{|g_n|^2 + |h_n|^2}.$$

Then u_n are positive subharmonic functions, and

$$\Delta u_n = 4(f_n^\#)^2$$

by direct computation. (This identity expresses the fact that the spherical metric has curvature 4). Fix arbitrary $r < 1$. Let

$$u_n = v_n + w_n$$

be the Riesz decomposition, where v_n are the least harmonic majorants of u_n in the disc $D(r)$, and w_n are Green potentials. As the Riesz measures of u_n and w_n in $D(r)$ are uniformly bounded, one can select a subsequence such that the corresponding potentials w_n will converge to some subharmonic function $w^* \not\equiv -\infty$. Let k_n be analytic functions without zeros in $D(r)$ such that $v_n = -\log|k_n|$. Then we have $f_n = g_n k_n / h_n k_n$, and

$$\log\sqrt{|g_n k_n|^2 + |h_n k_n|^2} = w_n \to w^*.$$

So the sequences $g_n k_n$ and $h_n k_n$ are uniformly bounded on compact subsets of $D(r)$ and by choosing another subsequence we obtain that $g_n k_n \to g^*$ and $h_n k_n \to h^*$, where g^* and h^* are holomorphic in $D(r)$, one of them is not identical to zero. Then $f_n \to g^*/h^*$ uniformly on compact subsets of

$$D(r) \backslash \{\text{common zeros of } g^*, h^*\}.$$

As $r < 1$ is arbitrary, we obtain the statement about quasinormality. Now, the number of common zeros of g^* and h^* cannot exceed p, (each common zero contributes πp to the Laplacian of w^*), so we have quasinormality of order p.

Remark. Simple examples show that a family can be quasinormal in the unit disc with an irregular point at zero, but the functions of the family are not uniformly spherically p-valent in any disc centered at 0.

Theorem 2. *Let f be a holomorphic function without zeros in the unit disc, which is spherically p-valent in the ring $A = \{1/2 < |z| < 1\}$.*

$$(1 - |z|^2) f^{\#}(z) \leq 9p \log 2 + 3 \log 2 + 2/\sqrt{e} \approx 6.24\, p + 3.31, \quad |z| < 1,$$

where $f^{\#}$ stands for the spherical derivative.

Remarks. 1. The example $f(z) = az$ which is univalent but may have large derivative at 0 shows that our additional assumption that f has no zeros is essential.
2. Our estimate is of interest only when $p \geq 1$ because it does not tend to zero when $p \to 0$. When $p < 1$ in fact no assumption about omitted zeros is necessary and the best possible estimate for this case, due to Duffresnoy, is given in [2, Theorem 6.1]
3. A theorem of Montel ([3, p.66]) shows that a quasinormal family of holomorphic functions omitting one value is in fact normal. It follows that under the assumptions of the Theorem there exists a bound for the derivative. Our purpose was to obtain an explicit bound.
4. A similar result with the same proof holds for holomorphic curves in n-dimensional projective space, omitting $n+1$ hyperplanes, the estimate depends on the omitted hyperplanes and dimension. This also applies to Duffresnoy's theorem mentioned in 2.

Proof. A standard argument reduces the problem to the case when f is continuous in the closed disc $\bar{\mathbf{U}}$. In view of conformal invariance it is enough to prove the estimate for $z = 0$. We also assume without loss of generality that

$$|f(0)| \leq 1. \tag{1}$$

Let h be the least harmonic majorant of $\log^+ |f|$ in $\mathbf{U}$. Then $h = -\log |f_0|$ for some holomorphic function without zeros in $\mathbf{U}$, and we set $f_1 = f f_0$, so that

$$f = f_1/f_0.$$

Now we define harmonic functions $u_j = \log |f_j|,\ j = 0, 1$ and notice that these functions are negative and their least common harmonic majorant is 0. We also put $v = u_0 \vee u_1$ (pointwise maximum) and $u = \log \sqrt{|f_0|^2 + |f_1|^2}$. Both functions u and v are subharmonic and satisfy

$$v \leq u \leq v + (\log 2)/2, \quad v \leq 0 \quad \text{and} \quad v(z) = 0, |z| = 1. \tag{2}$$

In addition we have

$$\Delta u = f^{\#}.$$

Thus applying Jensen's Formula to u we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta})\, d\theta + \frac{1}{\pi} \int_r^1 A(t) \frac{dt}{t} \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta})\, d\theta,$$

where $A(t)$ is the spherical area of the f-image of the disc $|z| \leq t$, and $r \in (0, 1)$ is a number to be specified later. Thus we can find a point z_0 with $|z_0| = r$, such that

$$u(z_0) \geq \int_{-\pi}^{\pi} u(e^{i\theta}) \frac{d\theta}{2\pi} - p \log(1/r),$$

where we used the assumption that $A(t) \leq p\pi$. Using the unequalities (2) we conclude that

$$v(z_0) \geq -p \log(1/r) - (\log 2)/2.$$

Thus for some $j \in \{0, 1\}$ we have

$$u_j(z_0) \geq -p \log(1/r) - (\log 2)/2,$$

and using the assumption (1), which is equivalent to $u_0(0) \geq u_1(0)$, and Harnack's inequality for negative harmonic function u_j, we obtain

$$u_0(0) \geq u_j(0) \geq \frac{1+r}{1-r} u_j(z_0) \geq -\frac{1+r}{1-r}(p \log(1/r) + (\log 2)/2).$$

One can optimize with respect to r, but let us just put $r = 1/2$:

$$u_0(0) \geq -3p \log 2 - 3(\log 2)/2 =: -C(p).$$

Lemma. *For negative harmonic functions w in the unit disc*

$$|\operatorname{grad} w(0)| \leq 2|w(0)|.$$

This follows from the Poisson formula. □

Now we consider two cases.

Case 1. $|f_1(0)| \geq |f_0(0)|^2$. This is equivalent to $u_1(0) \geq 2u_0(0)$. Thus by the Lemma $|\mathrm{grad}\, u_0(0)| \leq 2C(p)$ and $|\mathrm{grad}\, u_1(0)| \leq 4C(p)$. We have

$$f^{\#}(0) = \frac{|f_0 f_1|}{|f_0|^2 + |f_1|^2} \left| \frac{f_0'}{f_0} + \frac{f_1'}{f_1} \right| \leq \frac{1}{2}(|\mathrm{grad}\, u_0| + |\mathrm{grad}\, u_1|)$$

and $f^{\#}(0) \leq 3C(p)$ follows.

Case 2. $|f_1(0)| \leq |f_0(0)|^2$. In this case we write

$$f^{\#}(0) \leq \frac{|f_0 f_1|}{|f_0|^2} (|\mathrm{grad}\, u_0| + |\mathrm{grad}\, u_1|) \leq \left| \frac{f_1}{f_0} \right| |\mathrm{grad}\, u_0| + \frac{\sqrt{|f_1|}}{|f_0|} \sqrt{|f_1|} |\mathrm{grad}\, u_1|,$$

which gives in view of the Lemma the estimate

$$f^{\#}(0) \leq 2C(p) + \max_{0 \leq x \leq 1} (2\sqrt{x} \log(1/x)) = 2C(p) + 2/\sqrt{e}.$$

References

[1] W. Hayman, Multivalent Functions, Cambridge, 1994.

[2] W. Hayman, Meromorphic Functions, Oxford, 1964.

[3] P. Montel, Leçons sur les familles normales de fonctions analytiques at leurs applications, Paris, Gauthier-Villars, 1927.

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